Equality of the bulk and edge Hall conductances in a mobility gap

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Abstract: We consider the edge and bulk conductances for 2D quantum Hall systems in which the Fermi energy falls in a band where bulk states are localized. We show that the resulting quantities are equal, when appropriately defined. An appropriate definition of the edge conductance may be obtained through a suitable time averaging procedure or by including a contribution from states in the localized band. In a further result on the Harper Hamiltonian, we show that this contribution is essential. In an appendix we establish quantized plateaus for the conductance of systems which need not be translation ergodic.

1. Introduction

Two conductances, σ_B and σ_E, are associated to the Quantum Hall Effect (QHE), depending on whether the currents are ascribed to the bulk or to the edge. The equality $\sigma_B = \sigma_E$, suggested by Halperin's analysis [17] of the Laughlin argument [21], has been established in the context of an effective field theory description [14]. It was later derived in a microscopic treatment of the integral QHE [32,12,24] for the case that the Fermi energy lies in a spectral gap Δ of the single-particle Hamiltonian H_B. We prove this equality, by quite different means, in the more general setting that H_B exhibits Anderson localization in Δ —more precisely, dynamical localization (see (1.2) below). The result applies to Schrödinger operators which are random, but does not depend on that property. We therefore formulate the result for deterministic operators. The relation to recent work [7] will be discussed below.

The Bulk is represented by the lattice $\mathbb{Z}^2 \ni x = (x_1, x_2)$ with Hamiltonian $H_B = H_B^*$ on $\ell^2(\mathbb{Z}^2)$. We assume its matrix elements $H_B(x, x')$, $x, x' \in \mathbb{Z}^2$, to be of short range in the sense that

$$\sup_{x\in\mathbb{Z}^2} \sum_{x'\in\mathbb{Z}^2} |H_B(x,x')|\,(\mathrm{e}^{\mu|x-x'|} - 1) \;=:\; C_1 \;<\; \infty \tag{1.1}$$

for some $\mu > 0$, where $|x| = |x_1| + |x_2|$. Our hypothesis on the bounded open interval $\Delta \subset \mathbb{R}$ is that for some $\nu \geq 0$

$$\sup_{g \in B_1(\Delta)} \sum_{x,x' \in \mathbb{Z}^2} |g(H_B)(x,x')| \, (1+|x|)^{-\nu} \mathrm{e}^{\mu|x-x'|} \;=:\; C_2 \;<\; \infty \tag{1.2}$$

where $B_1(\Delta)$ denotes the set of Borel measurable functions g which are constant in $\{\lambda | \lambda < \Delta\}$ and in $\{\lambda | \lambda > \Delta\}$ with $|g(x)| \leq 1$ for every x.

In particular C_2 is a bound when g is of the form $g_t(\lambda) = \mathrm{e}^{-\mathrm{i}t\lambda} E_\Delta(\lambda)$ and the supremum is over $t \in \mathbb{R}$, which is a statement of dynamical localization. By the RAGE theorem this implies that the spectrum of H_B is pure point in Δ (see [20] or [10, Theorem 9.21] for details). We denote the corresponding eigenprojections by $E_{\{\lambda\}}(H_B)$ for $\lambda \in \mathcal{E}_\Delta$, the set of eigenvalues $\lambda \in \Delta$. We assume that no eigenvalue in $\mathcal{E}_\Delta$ is infinitely degenerate,

$$\dim E_{\{\lambda\}}(H_B) \;<\; \infty\,, \quad \lambda \in \mathcal{E}_\Delta\,. \tag{1.3}$$

The validity of these assumptions is discussed below (but see also [2,4]).

The zero temperature bulk Hall conductance at Fermi energy λ is defined by the Kubo-Středa formula [5]

$$\sigma_B(\lambda) \;=\; -\mathrm{i}\,\mathrm{tr}\, P_\lambda \left[\, [P_\lambda, \Lambda_1]\,, [P_\lambda, \Lambda_2]\,\right]\,, \tag{1.4}$$

where $P_\lambda = E_{(-\infty,\lambda)}(H_B)$ and $\Lambda_i(x)$ is the characteristic function of

$$\left\{ x = (x_1, x_2) \in \mathbb{Z}^2 \mid x_i < 0 \right\}\,.$$

Under the above assumptions $\sigma_B(\lambda)$ is well-defined for $\lambda \in \Delta$, but independent thereof, i.e., it shows a plateau. (This result, first proved in [6], is strengthened here in an appendix, since we do not assume translation covariance or ergodicity of the Schrödinger operator. We also show the integrality of $2\pi\sigma_B$ therein, though it is not needed in the sequel.) We remark that (1.3) is essential for a plateau: for the Landau Hamiltonian (though defined on the continuum rather than on the lattice) eqs. (1.1, 1.2) hold if properly interpreted, but (1.3) fails in an interval containing a Landau level, where indeed $\sigma_B(\lambda)$ jumps.

The sample with an Edge is modeled as a half-plane $\mathbb{Z} \times \mathbb{Z}_a$, where $\mathbb{Z}_a = \{n \in \mathbb{Z} \mid n \geq -a\}$, with the height $-a$ of the edge eventually tending to $-\infty$. The Hamiltonian $H_a = H_a^*$ on $\ell^2(\mathbb{Z} \times \mathbb{Z}_a)$ is obtained by restriction of H_B under some largely arbitrary boundary condition. More precisely, we assume that

$$E_a = \mathcal{J}_a H_a - H_B \mathcal{J}_a \;:\; \ell^2(\mathbb{Z} \times \mathbb{Z}_a) \to \ell^2(\mathbb{Z}^2) \tag{1.5}$$

satisfies

$$\sup_{x \in \mathbb{Z}^2} \sum_{x' \in \mathbb{Z} \times \mathbb{Z}_a} |E_a(x,x')| \, \mathrm{e}^{\mu\left(|x_2+a| + |x_1 - x_1'|\right)} \;\leq\; C_3 \;<\; \infty\,, \tag{1.6}$$

where $\mathcal{J}_a : \ell^2(\mathbb{Z} \times \mathbb{Z}_a) \to \ell^2(\mathbb{Z}^2)$ denotes extension by 0. For instance with Dirichlet boundary conditions, $H_a = \mathcal{J}_a^* H_B \mathcal{J}_a$, we have $E_a = (\mathcal{J}_a \mathcal{J}_a^* - 1) H_B \mathcal{J}_a$, i.e.,

$$E_a(x,x') \;=\; \begin{cases} -H_B(x,x')\,, & x_2 < -a\,, \\ 0\,, & x_2 \geq -a\,, \end{cases}$$

whence (1.6) follows from (1.1). We remark that eq. (1.1) is inherited by H_a with a constant C_1 that is uniform in a, but not so for eq. (1.2) as a rule.

The definition of the edge Hall conductance requires some preparation. The current operator across the line $x_1 = 0$ is $-\mathrm{i}\,[H_a, \Lambda_1]$. Matters are simpler if we temporarily assume that Δ is a gap for H_B, i.e., if $\sigma(H_B) \cap \Delta = \emptyset$, in which case one may set [32]

$$\sigma_E \ := \ -\mathrm{i}\,\mathrm{tr}\,\rho'(H_a)\,[H_a, \Lambda_1] \ , \tag{1.7}$$

where $\rho \in C^\infty(\mathbb{R})$ satisfies

$$\rho(\lambda) \ = \ \begin{cases} 1\,, & \lambda < \Delta\,, \\ 0\,, & \lambda > \Delta\,. \end{cases} \tag{1.8}$$

The heuristic motivation for (1.7) is as follows. We interpret $\rho(H_a)$ as the 1-particle density matrix of a stationary quantum state. Though some current is flowing near the edge we should discard it, as it is supposed to be canceled by current flowing at an opposite edge located at $x_2 = +\infty$. If the chemical potential is now lowered by δ at the first edge, but not at the second, a net current

$$I \ = \ -\mathrm{i}\,\mathrm{tr}\,((\rho(H_a + \delta) - \rho(H_a))\,[H_a, \Lambda_1]) \ = \ -\mathrm{i} \int_0^\delta \mathrm{d}t\,\mathrm{tr}\,\rho'(H_a - t)\,[H_a, \Lambda_1]$$

is flowing. Since σ_E is independent of ρ as long as it conforms with (1.8), see [32] and Theorem 1 below, it is indeed the conductance $\sigma_E = I/\delta$ for sufficiently small δ.

The operator in (1.7) is trace class essentially because $\mathrm{i}\,[H, \Lambda_1]$ is relevant only on (single-particle) states near $x_1 = 0$, and $\rho'(H_a)$ only near the edge $x_2 = -a$, so that the intersection of the two strips is compact. In the situation (1.2) considered in this paper the operator appearing in (1.7) is not trace class, since the bulk operator may have spectrum in Δ, which can cause the above stated property to fail for $\rho'(H_a)$. In search of a proper definition of σ_E, we consider only the current flowing across the line $x_1 = 0$ within a finite window $-a \le x_2 < 0$ next to the edge. This amounts to modifying the current operator to be

$$-\frac{\mathrm{i}}{2}\,(\Lambda_2\,[H_a, \Lambda_1] + [H_a, \Lambda_1]\,\Lambda_2) \ = \ -\frac{\mathrm{i}}{2}\,\{\,[H_a, \Lambda_1]\,, \Lambda_2\} \ , \tag{1.9}$$

with which one may be tempted to use

$$\lim_{a\to\infty} -\frac{\mathrm{i}}{2}\,\mathrm{tr}\,\rho'(H_a)\,\{\,[H_a, \Lambda_1]\,, \Lambda_2\} \tag{1.10}$$

as a definition for σ_E. Though we show that this limit exists, it is not the physically correct choice. We may in fact expect that the dynamics of $\mathrm{e}^{-\mathrm{i}tH_a}$ acting on states supported far away from the edge resembles for quite some time the dynamics generated by H_B. Being bound states or, more likely, resonances, such states may carry persistent currents (whence the operator in (1.7) is not trace class), but no or little net current across the line $x_1 = 0$. This cancelation is the rationale for ignoring the part $x_2 \ge 0$ of the line $x_1 = 0$ by means of the cutoff Λ_2 in (1.9), however the cancelation is not achieved on states located near

the end point $x = (0,0)$. In the limit $a \to \infty$ we pretend these states are bound, which yields the contribution missed by (1.10):

$$-\frac{\mathrm{i}}{2}\left(\psi_\lambda, \{\,[H_B, \Lambda_1]\,, 1-\Lambda_2\}\,\psi_\lambda\right) \;=\; \operatorname{Im}\left(\psi_\lambda, \Lambda_1 H_B \Lambda_2 \psi_\lambda\right)\;, \tag{1.11}$$

from each bound state ψ_λ of H_B, with corresponding energy $\lambda \in \mathcal{E}_\Delta$. We incorporate them with weight $\rho'(\lambda)$ in our definition of the edge conductance:

$$\sigma_E^{(1)} \;:=\; \lim_{a\to\infty} -\frac{\mathrm{i}}{2}\operatorname{tr}\rho'(H_a)\,\{\,[H_a, \Lambda_1]\,, \Lambda_2\} + \sum_{\lambda\in\mathcal{E}_\Delta}\rho'(\lambda)\operatorname{Im}\operatorname{tr}E_{\{\lambda\}}\Lambda_1 H_B \Lambda_2 E_{\{\lambda\}}\;. \tag{1.12}$$

We will show that the sum on the r.h.s. is absolutely convergent, and its physical meaning will be further discussed at the end of the Introduction. We will also show it to be non-zero on average for the Harper Hamiltonian with an i.i.d. random potential in Theorem 3.

The terms of this sum involve H_B, though the few states for which they are sizeable are supported near $x = (0,0)$ and hence far from the edge $x_2 = -a$. Since the mere appearance of H_B in the definition of an edge property may be objectionable, we present an alternative. The basic fact that the net current of a bound state is zero,

$$-\mathrm{i}\left(\psi_\lambda, [H_B, \Lambda_1]\,\psi_\lambda\right) \;=\; 0\;, \tag{1.13}$$

can be preserved by the regularization provided the spatial cutoff Λ_2 is time averaged. In fact, let

$$A_{T,a}(X) \;=\; \frac{1}{T}\int_0^T \mathrm{e}^{\mathrm{i}H_a t} X \mathrm{e}^{-\mathrm{i}H_a t}\mathrm{d}t \tag{1.14}$$

be the time average over $[0,T]$ of a (bounded) operator X with respect to the Heisenberg evolution generated by H_a, with a finite or $a = B$. If a limit $\Lambda_2^\infty = \lim_{T\to\infty} A_{T,B}(\Lambda_2)$ were to exist, it would commute with H_B so that

$$-\frac{\mathrm{i}}{2}\left(\psi_\lambda, \{\,[H_B, \Lambda_1]\,, \Lambda_2^\infty\}\,\psi_\lambda\right) \;=\; 0\;.$$

This motivates our second definition,

$$\sigma_E^{(2)} \;:=\lim_{T\to\infty}\lim_{a\to\infty} -\frac{\mathrm{i}}{2}\operatorname{tr}\rho'(H_a)\,\{\,[H_a, \Lambda_1]\,, A_{T,a}(\Lambda_2)\}\;. \tag{1.15}$$

The two definitions allow for the following result.

Theorem 1. *Under the assumptions (1.1, 1.2, 1.3, 1.6, 1.8) the sum in (1.12) is absolutely convergent, the limits there and in (1.15) exist, and*

$$\sigma_E^{(1)} \;=\; \sigma_E^{(2)} \;=\; \sigma_B\;.$$

In particular (1.12, 1.15) depend neither on the choice of ρ nor on that of E_a.

Remark 1. i.) The hypotheses (1.1, 1.2) hold almost surely for ergodic Schrödinger operators whose Green's function $G(x,x';z) = (H_B - z)^{-1}(x,x')$ satisfies a moment condition [3] of the form

$$\sup_{E\in\Delta} \limsup_{\eta\to 0} \mathbb{E}\left(|G(x,x';E+\mathrm{i}\eta)|^s\right) \ \le\ C\mathrm{e}^{-\mu|x-x'|} \tag{1.16}$$

for some $s<1$. The implication is through the dynamical localization bound

$$\mathbb{E}\left(\sup_{g\in B_1(\Delta)} |g(H_B)(x,x')|\right) \ \le\ C\mathrm{e}^{-\mu|x-x'|}\ , \tag{1.17}$$

although (1.2) has also been obtained by different means, e.g., [16]. The implication (1.16) $\Rightarrow$ (1.17) was proved in [1] (see also [2,11,4]). The bound (1.17) may be better known for $\operatorname{supp} g \subset \Delta$, but is true as stated since it also holds [6, 2] for the projections $g(H_B) = P_\lambda$, $P_\lambda^\perp = 1 - P_\lambda$, $(\lambda \in \Delta)$.

ii.) Condition (1.3), in fact simple spectrum, follows form the arguments in [34], at least for operators with nearest neighbor hopping, $H_B(x,y) = 0$ if $|x - y| > 1$.

iii.) When $\sigma(H_B)\cap\Delta = \emptyset$, the operator appearing in (1.7) is known to be trace class. In this case, the conductance $\sigma_E^{(1)} = \sigma_E^{(2)}$ defined here coincides with σ_E defined in (1.7). This statement follows from Theorem 1 and the known equality $\sigma_E = \sigma_B$ [32,12], but can also be seen directly. For completeness, we include a proof of this fact in Section 2 below.

A point of view which combines both definitions of the edge conductance is expressed by the following result.

Theorem 2. *Under the assumptions of Theorem 1,*

$$\begin{aligned} &\lim_{a\to\infty} -\frac{\mathrm{i}}{2}\operatorname{tr}\rho'(H_a)\left\{\,[H_a,\Lambda_1]\ ,\ \Lambda_{2;a}(t)\right\} \\ &\qquad =\ \sigma_B\ +\ \sum_{\lambda\in\mathcal{E}_\Delta}\rho'(\lambda)\operatorname{Im}\operatorname{tr}E_{\{\lambda\}}\,[H_B,\Lambda_1]\,\mathrm{e}^{\mathrm{i}H_Bt}\Lambda_2\mathrm{e}^{-\mathrm{i}H_Bt}E_{\{\lambda\}}\ , \end{aligned} \tag{1.18}$$

with $\Lambda_{2;a}(t) = \mathrm{e}^{\mathrm{i}H_at}\Lambda_2\mathrm{e}^{-\mathrm{i}H_at}$.

In particular, this reduces to $\sigma_E^{(1)} = \sigma_B$ for $t=0$ by (1.11, 1.12). On the other hand, $\sigma_E^{(2)} = \sigma_B$ results, as we will show, from the time average of (1.18).

A recent preprint [7] contains results which are topically related to but substantially different from those presented here. In that work, two contiguous media are modeled by positing a potential of the form $U(x_1,x_2) = V_0(x_2)\chi(x_2 < 0) + V(x_1,x_2)\chi(x_2 \ge 0)$ (in our notation), where V_0 is independent of x_1. The role of V is that of a bulk potential, and that of V_0 as of a wall, provided it is large. The kinetic term is given by the Landau Hamiltonian on the continuum $L^2(\mathbb{R}^2)$, whose unperturbed spectrum is the familiar set $(2\mathbb{N}+1)B$, with B the magnitude of the constant magnetic field. A result is the following: if model (a), with $V_0 = 0$, exhibits localization in $\Delta \subset [(2N-1)B, (2N+1)B]$ for some positive integer N, and hence $\sigma_E = 0$, then model (b), with $V_0(x_2) \ge (2N+1)B$, has $2\pi\sigma_E = N$. The result is established by showing that the difference between

$2\pi\sigma_E$ in cases (b) and (a) is independent of V, and equals N if $V = 0$, the two models then being solvable thanks to the translation invariance w.r.t. x_1.

In comparison to our work, the following features may be noted:

i.) The localization assumption on the reference model (a) is made for a system which has itself an interface. (Our eq. (1.2) concerns a bulk model serving as reference.)

ii.) The validity of that assumption is limited to small V, because the interface of (a) will otherwise produce extended edge states with energies in Δ. The result $\sigma_E = \sigma_B$ thus applies to perturbations of the free Landau Hamiltonian of size $\lesssim B$. (Our comparison $\sigma_E = \sigma_B$ does not require either side to be explicitly computable.)

iii.) The definition of σ_E for (b) depends on eigenstates in Δ of (a), like our $\sigma_E^{(1)}$, but not $\sigma_E^{(2)}$.

A model without bulk potential, but allowing interactions between diluted particles, was studied from a related perspective in [25].

In (1.11, 1.12) we argued that the limit (1.10) is not identical to σ_B. To indeed prove this, we show that the sum on the right hand side of (1.12) does not vanish for the Harper Hamiltonian with i.i.d. Cauchy randomness on the diagonal.

The Harper Hamiltonian models the hopping of a tightly bound charged particle in a uniform magnetic field. The hopping terms $H(x, x')$ are zero except for nearest neighbor pairs, for which they are of modulus one,

$$|H(x,x')| \;=\; \begin{cases} 0\,, & |x-x'| \neq 1\,, \\ 1\,, & |x-x'| = 1\,, \end{cases} \tag{1.19}$$

where the non-zero matrix elements are interpreted as

$$H(x,x') \;=\; \mathrm{e}^{\mathrm{i}\int_{x'}^{x} \mathbf{A}(y)\cdot\,\mathrm{d}^1 y}\,,$$

with $\mathbf{A}$ the magnetic vector potential and the line integral computed along the bond connecting x, x'. The magnetic flux through any region $D \subset \mathbb{R}^2$ is

$$\int_D B(x)\mathrm{d}^2 x \;=\; \int_{\partial D} \mathbf{A}(y)\cdot \mathrm{d}^1 y\,,$$

so, for a uniform field, the flux is proportional to the area

$$\int_{\partial D} \mathbf{A}(y)\cdot \mathrm{d}^1 y \;=\; \phi |D|\,.$$

Thus, we require that

$$H(x^{(1)},x^{(4)})H(x^{(4)},x^{(3)})H(x^{(3)},x^{(2)})H(x^{(2)},x^{(1)}) \;=\; \mathrm{e}^{\mathrm{i}\int_{\partial P}\mathbf{A}(y)\cdot\mathrm{d}^1 y} \;=\; \mathrm{e}^{\mathrm{i}\phi} \tag{1.20}$$

where $x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}$ are the vertices of a plaquette P, listed in counter clockwise order and ϕ is the flux through any plaquette.

There are many choices of nearest neighbor hopping terms which satisfy (1.19) and (1.20), all interrelated by gauge transformations. For our purposes, it suffices to fix a gauge and take

$$H_\phi(x,x') \ := \ \begin{cases} 1\,, & x = x' \pm e_1\,, \\ \mathrm{e}^{\mathrm{i}\phi x_1}\,, & x = x' + e_2\,, \\ \mathrm{e}^{-\mathrm{i}\phi x_1}\,, & x = x' - e_2\,, \end{cases} \tag{1.21}$$

with $e_1 = (1,0)$ and $e_2 = (0,1)$ the lattice generators. This choice of H_ϕ comes from representing the constant field $B = \phi$ via the vector potential $\mathbf{A} = \phi(0, x_1)$. We note that the bulk and edge Hall conductances are gauge invariant quantities, so Theorem 3 stated below holds for any other choice of H_ϕ. We refer the reader to ref. [26] and references therein for further discussion of the Harper Hamiltonian.

To guarantee localized spectrum, we consider a bulk Hamiltonian which consists of H_ϕ plus a diagonal random potential,

$$H_B \ = \ H_\phi + \alpha V$$

where $V\psi(x) \ = \ V(x)\psi(x)$ and $V(x)$, $x \in \mathbb{Z}^2$ are independent identically distributed Cauchy random variables. Here α is a coupling parameter (the "disorder strength") and "Cauchy" signifies that the distribution of $v = V(x)$ is

$$\frac{1}{\pi}\frac{1}{1+v^2}\mathrm{d}v\ .$$

We use Cauchy variables because it is possible to calculate certain quantities explicitly for such variables: $\mathbb{E}\left(f(v)\right) = f(\mathrm{i})$ for a function f having a bounded analytic continuation to the upper half plane.

It is clear that H_B is short range, i.e., (1.1) holds. For simplicity we consider H_a which are defined via a non-random boundary conditions, i.e., the operators E_a appearing in (1.5) do not depend on the random couplings $V(x)$. We then have the following result.

Theorem 3. *For $H_B,\ H_a$ as above, there is $j_B \in C^\infty$ such that*

$$\mathbb{E}\left(-\frac{\mathrm{i}}{2}\lim_{a\to\infty} \operatorname{tr}\rho'(H_a)\left\{\left[H_a, \Lambda_1\right], \Lambda_2\right\}\right) \ = \ -\int \rho'(\lambda) j_B(\lambda)\mathrm{d}\lambda\ , \tag{1.22}$$

whenever $\rho' \in C_0^\infty(\mathbb{R})$. The expectation is well defined and may be interchanged with the limit. Furthermore, $j_B(\lambda)$ has the following asymptotic behavior

$$j_B(\lambda) \ = \ -\frac{4|\alpha|}{\pi}\sin(\phi)(\cos(\phi)+1)\lambda^{-5} \ + \ \mathcal{O}(\lambda^{-6})\ , \quad |\lambda| \to \infty\ . \tag{1.23}$$

The result is relevant in relation to (1.12) since it has in fact been shown that (1.2) holds for H_B at large energies:

Theorem ([1]). *There is $E_0(\alpha)$ such that* (1.17) *holds for H_B and $\Delta = \Delta_\pm$ with $\Delta_- = (-\infty, -E_0(\alpha)]$ and $\Delta_+ = [E_0(\alpha), \infty)$. Hence* (1.2) *holds almost surely.*

Remark 2. i.) For any $\alpha \neq 0$ the spectrum of H_B is (almost surely) the entire real line, so the eigenvalues of H_B in $\Delta_\pm$ make up a (random) dense subset which we denote $\mathcal{E}_{\Delta_\pm}$. In fact, this pure point spectrum is almost surely simple, as can be shown using the methods in [34]. ii.) For sufficiently large α we have $E_0(\alpha) = 0$, i.e., the spectrum is completely localized. iii.) Localization also holds inside the spectral gaps of H_ϕ, for small α, via the methods in [1,4].

The mentioned result implies $\sigma_B(\lambda) = 0$ for $\lambda \in \Delta_\pm$, because σ_B is insensitive to λ in that range and $P_\lambda \to 1$ or 0 as $\lambda \to \infty$ or $-\infty$, respectively. Thus for ρ as in (1.8) with $\operatorname{supp}\rho' \subset \Delta_\pm$ we have $\sigma_E^{(1)} = 0$ by Theorem 1. On the other hand, for the first term on the r.h.s. of (1.12), $J_B(\rho)$, we have by Theorem 3

$$\mathbb{E}\left(J_B(\rho)\right) \;=\; \frac{4|\alpha|}{\pi}\sin(\phi)(\cos(\phi)+1)\int_{|\lambda|\geq E_0(\alpha)}\rho'(\lambda)\lambda^{-5}\mathrm{d}\lambda \;+\; \mathcal{O}\left(\lambda_0^{-6}\right) \tag{1.24}$$

as $\lambda_0 = \inf\{|\lambda| | \lambda \in \operatorname{supp}\rho'\} \to \infty$. Clearly the right hand side can be non-zero for appropriately chosen ρ, and the same then holds for the expectation of the last term in (1.12).

The definitions (1.12, 1.15) may be related, heuristically, to concepts from classical electro-magnetism of material media [31]. There the macroscopic (or average) current is split as $\boldsymbol{j}_f + \partial\boldsymbol{P}/\partial t + \mathbf{rot}\, M$ into free, polarization, and magnetization currents. (The magnetization M is a scalar in two dimensions.) The distinction depends on the existence of units (free electrons, atoms, molecules, ...) each with conserved charge, whose current densities are effectively of the form

$$\boldsymbol{j}(\boldsymbol{x},t) \;=\; q\dot{\boldsymbol{r}}(t)\delta(\boldsymbol{x}-\boldsymbol{r}(t)) + \frac{\partial}{\partial t}\delta(\boldsymbol{x}-\boldsymbol{r}(t))\boldsymbol{p}(t) + \mathbf{rot}\left(\delta(\boldsymbol{x}-\boldsymbol{r}(t))m(t)\right)\;, \tag{1.25}$$

where $q, \boldsymbol{p}(t), m(t)$ are the unit's charge and electric/magnetic moments respectively. The macroscopic quantities emerge as a weak limit of the microscopic ones

$$\sum_k \delta(\boldsymbol{x}-\boldsymbol{r}_k(t))\left\{\begin{matrix} q_k\dot{\boldsymbol{r}}_k(t) \\ \boldsymbol{p}_k(t) \\ m_k(t)\end{matrix}\right\} \;\rightharpoonup\; \left\{\begin{matrix} \boldsymbol{j}_f(\boldsymbol{x},t) \\ \boldsymbol{P}(\boldsymbol{x},t) \\ M(\boldsymbol{x},t)\end{matrix}\right\}\;,$$

or more precisely after integration against compactly supported test functions which vary slowly over the interatomic distance. The microscopic current across the portion $x_2 \leq 0$ of the line $x_1 = 0$ is then

$$\begin{aligned} I \;&=\; -\sum_k \int \mathrm{d}^2\boldsymbol{x}\Lambda'(x_1)\Lambda(x_2)\boldsymbol{j}_{k,1}(\boldsymbol{x},t) \\ &=\; -\int \mathrm{d}^2\boldsymbol{x}\Lambda'(x_1)\Lambda(x_2)\left[\boldsymbol{j}_{f,1}(\boldsymbol{x},t) + \frac{\partial\boldsymbol{P}_1}{\partial t}(\boldsymbol{x},t)\right] \\ &\quad\; + \int \mathrm{d}^2\boldsymbol{x}\Lambda'(x_1)\Lambda'(x_2)M(\boldsymbol{x},t)\;. \end{aligned} \tag{1.26}$$

The derivation assumes that Λ is smooth over interatomic distances. The last term in (1.26) comes from the corresponding term in (1.25), which is $\partial_2\delta(\boldsymbol{x} - \boldsymbol{r}_k(t))m_k(t)$. It cannot be replaced by adding $(\mathbf{rot}M)_1 = \partial_2 M$ within the square

brackets, which would correspond to the macroscopic current. In fact, it differs from that by a boundary term, which would vanish if $\Lambda(x_2)$ were compactly supported. Let now the macroscopic fields be stationary and slowly varying on the scale of Λ'. In the QHE we expect that the (free) edge currents are located near the edge, so that (1.26) becomes

$$I \;=\; \int_{x_1=0} \mathrm{d}x_2 \boldsymbol{j}_{f,1}(\boldsymbol{x}) \;+\; M(0) \; .$$

When $M(0)$ is subtracted from the l.h.s., we obtain an expression for the edge current, which is the role of the second term in (1.12). In this analogy the definition (1.15) corresponds to replacing $\Lambda(x_2)$ in the first line of (1.26) by $\Lambda(\boldsymbol{e}_2 \cdot \boldsymbol{r}_{k,T})$ where $\boldsymbol{r}_{k,T}$ is the time average of $\boldsymbol{r}_k(t)$. Then the last term no longer arises.

The above discussion neglects the weighting $\rho'(\lambda)$ of energies in (1.12). This will be remedied in the following heuristic argument in support of $\sigma_B = \sigma_E$. In a finite sample of volume V the Středa relation [35] asserts

$$\frac{\partial N}{\partial \phi} \;\cong\; \sigma_B V \; , \tag{1.27}$$

where N is the total charge of carriers, i.e., $N = \operatorname{tr} \rho(H_V)$ in the situation considered here. For the total magnetization M we have

$$-\frac{\partial M}{\partial \mu} \;=\; \operatorname{tr} \rho'(H_V) \frac{\partial H_V}{\partial \phi} \; , \tag{1.28}$$

where μ is the chemical potential, as can be seen from the Maxwell relation [15]

$$-\frac{\partial M}{\partial \mu} \;=\; \frac{\partial N}{\partial \phi} \; . \tag{1.29}$$

To compute $\partial H_V / \partial \phi$ we use a gauge equivalent to (1.21), with trivial phases along bonds in direction $\boldsymbol{e}_2$, and obtain for (1.28)

$$-\frac{1}{V}\frac{\partial M}{\partial \mu} \;=\; \frac{1}{V}\frac{\mathrm{i}}{2} \operatorname{tr} \rho'(H_V) \left\{ [H_V, X_1] \; , \; X_2 \right\} \; .$$

By (1.27, 1.29) this quantity is formally σ_B. To relate it to $\sigma_E^{(1)}$ it should be noted that the total magnetization is <u>not</u> the integral of the bulk magnetization, even in the thermodynamic limit. For instance, for classical, spinless particles M vanishes [22], but consists [27] of a diamagnetic, bulk contribution and a an opposite contribution from states close to the edge. These two contributions (in reverse order) may be identified in the quantum mechanical context with the two terms of (1.12). In this example, the expected edge term is negative for $\phi > 0$. This should also emerge from (1.24) when $\sup \operatorname{supp} \rho' \to -\infty$, and it does if one also takes into account that $-H$ is the counterpart to the continuum Hamiltonian.

In Section 2 we will present the main steps in the proof of Theorems 1 and 2, with details supplied in Section 3. The proof of Theorem 3 will be given in Section 4. The appendix is about properties of σ_B.

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2. Outline of the proof

A reasonable first step is to make sure that the traces in (1.12, 1.15) are well-defined. We will show this for

$$\sigma_E(a,t) \;:=\; -\mathrm{i}\,\mathrm{tr}\,\rho'(H_a)\,[H_a,\Lambda_1]\,\Lambda_{2;a}(t)\;, \tag{2.1}$$

with $\Lambda_{2;a}(t) = \mathrm{e}^{\mathrm{i}tH_a}\Lambda_2\mathrm{e}^{-\mathrm{i}tH_a}$, by proving that $\mathrm{i}\,[H_a,\Lambda_1]\,\Lambda_{2;a}(t) \in \mathfrak{I}_1$ in Lemma 5. Here, $\mathfrak{I}_1$ denotes the ideal of trace class operators, and we denote the trace norm by $\|\cdot\|_1$. Then

$$\begin{aligned}\overline{\sigma_E(a,t)} \;&=\; -\mathrm{i}\,\mathrm{tr}\,\Lambda_{2;a}(t)\,[H_a,\Lambda_1]\,\rho'(H_a)\\ &=\; -\mathrm{i}\,\mathrm{tr}\,\rho'(H_a)\Lambda_{2;a}(t)\,[H_a,\Lambda_1]\;,\end{aligned}$$

where we used that

$$\mathrm{tr}\,AB \;=\; \mathrm{tr}\,BA \text{ if } AB\;,\; BA \;\in\; \mathfrak{I}_1\;, \tag{2.2}$$

e.g., [33, Corollary 3.8]. The definition (1.15) then reads

$$\sigma_E^{(2)} \;=\; \lim_{T\to\infty}\lim_{a\to\infty}\frac{1}{T}\int_0^T \mathrm{d}t\,\mathrm{Re}\,\sigma_E(a,t)\;. \tag{2.3}$$

By the argument given in the Introduction, the trace norm of the operator in (2.1) diverges as $a\to\infty$. To see that its trace nevertheless converges we subtract from it an operator $Z(a,t)\in\mathfrak{I}_1$, to be specified below, with $\mathrm{tr}\,Z(a,t)=0$, implying

$$\sigma_E(a,t) \;=\; -\mathrm{i}\,\mathrm{tr}\left(\rho'(H_a)\,[H_a,\Lambda_1]\,\Lambda_{2;a}(t) - Z(a,t)\right)\;. \tag{2.4}$$

The idea, of course, is to choose $Z(a,t)$ so that

$$\sup_a \left\|\rho'(H_a)\,[H_a,\Lambda_1]\,\Lambda_{2;a}(t) - Z(a,t)\right\|_1 \;<\; \infty\;. \tag{2.5}$$

An operator of zero trace is $[\rho(H_a),\Lambda_1]\,\Lambda_2$; it is trace class (see Lemma 5) and its trace, computed in the position basis, is seen to vanish. Though it does not quite suffice for (2.5), we consider it since $[\rho(H_a),\Lambda_1]$ and $\rho'(H_a)\,[H_a,\Lambda_1]$ are closely related: From the Helffer-Sjöstrand representations (see Section 3 for details)

$$\rho(H_a) \;=\; \frac{1}{2\pi}\int \mathrm{d}m(z)\partial_{\bar z}\rho(z)R(z) \tag{2.6a}$$

$$\rho'(H_a) \;=\; -\frac{1}{2\pi}\int \mathrm{d}m(z)\partial_{\bar z}\rho(z)R(z)^2\;, \tag{2.6b}$$

with $R(z) = (H_a - z)^{-1}$, we obtain

$$[\rho(H_a),\Lambda_1] \;=\; -\frac{1}{2\pi}\int \mathrm{d}m(z)\partial_{\bar z}\rho(z)R(z)\,[H_a,\Lambda_1]\,R(z) \tag{2.7a}$$

$$\rho'(H_a)\,[H_a,\Lambda_1] \;=\; -\frac{1}{2\pi}\int \mathrm{d}m(z)\partial_{\bar z}\rho(z)R(z)^2\,[H_a,\Lambda_1]\;. \tag{2.7b}$$

The two expressions, multiplied from the right by Λ_2, respectively by $\Lambda_{2;a}(t)$ as in (2.1), would have an even more similar structure if in the second a resolvent could be moved to the right. This can be achieved under the trace by setting

$$\begin{aligned} Z(a,t) \;&=\; [\rho(H_a), \Lambda_1]\,\Lambda_2 \\ &- \frac{1}{2\pi}\int \mathrm{d}m(z)\partial_{\bar z}\rho(z)R(z)\left(R(z)\,[H_a,\Lambda_1]\,\Lambda_{2;a}(t) - [H_a,\Lambda_1]\,\Lambda_{2;a}(t)R(z)\right)\;, \end{aligned} \tag{2.8}$$

for which $\mathrm{tr}\, Z(a,t) = 0$. Then (2.4) reads $\sigma_E(a,t) \;=\; \mathrm{tr}\,\Sigma_a(t)$ with

$$\mathrm{i}\Sigma_a(t) \;:=\; \overbrace{-\,[\rho(H_a),\Lambda_1]\,\Lambda_2}^{\mathrm{i}\widetilde{\Sigma}'_a} \tag{2.9}$$
$$+\;\underbrace{-\frac{1}{2\pi}\int \mathrm{d}m(z)\partial_{\bar z}\rho(z)R(z)\,[H_a,\Lambda_1]\,\Lambda_{2;a}(t)R(z)}_{\mathrm{i}\widetilde{\Sigma}''_a(t)}$$

$$=\;\overbrace{[\rho(H_a),\Lambda_1]\,(\Lambda_{2;a}(t) - \Lambda_2)}^{\mathrm{i}\Sigma'_a(t)} \tag{2.10}$$
$$+\;\underbrace{-\frac{1}{2\pi}\int \mathrm{d}m(z)\partial_{\bar z}\rho(z)R(z)\,[H_a,\Lambda_1]\,R(z)\,[H_a,\Lambda_{2;a}(t)]\,R(z),}_{\mathrm{i}\Sigma''_a(t)}$$

where, to obtain the last expression, (2.7a) multiplied by $\Lambda_{2;a}(t)$ has been added and subtracted, and $[R(z),\Lambda_{2;a}(t)] = -R(z)\,[H_a,\Lambda_{2;a}(t)]\,R(z)$ has been used. We remark that equality of (2.9) and (2.10) also holds for H_B, i.e., if we replace H_a by H_B and $\Lambda_{2;a}(t)$ by $\Lambda_{2;B}(t) = \mathrm{e}^{\mathrm{i}H_Bt}\Lambda_2\mathrm{e}^{-\mathrm{i}H_Bt}$, and set $R(z) = (H_B - z)^{-1}$.

We will show

$$\sigma_E(a,t) \;=\; \mathrm{tr}\,\Sigma_a(t) \;\xrightarrow[a\to\infty]{}\; \mathrm{tr}\,\Sigma_B(t) \tag{2.11}$$

and, incidentally, (2.5) by establishing:

Lemma 1. *Under assumptions (1.1, 1.6), but without making use of (1.2, 1.3, 1.8), we have for* $\rho' \in C_0^\infty(\mathbb{R})$

$$\|\mathcal{J}_a\Sigma'_a(t)\mathcal{J}_a^* - \Sigma'_B(t)\|_1 \;\xrightarrow[a\to\infty]{}\; 0\;, \tag{2.12}$$
$$\|\mathcal{J}_a\Sigma''_a(t)\mathcal{J}_a^* - \Sigma''_B(t)\|_1 \;\xrightarrow[a\to\infty]{}\; 0 \tag{2.13}$$

uniformly for t in a compact interval.

Note that the replacement $A \mapsto \mathcal{J}_aA\mathcal{J}_a^*$ simply extends by zero an operator on $\ell^2(\mathbb{Z}\times\mathbb{Z}_a)$ to one on $\ell^2(\mathbb{Z}^2)$. In particular $\|\mathcal{J}_aA\mathcal{J}_a^*\|_1 = \|A\|_1$ and $\mathrm{tr}\,\mathcal{J}_aA\mathcal{J}_a^* = \mathrm{tr}\,A$.

For the rest of this section on we shall only be concerned with Bulk quantities like $\mathrm{tr}\,\Sigma_B(t)$. By (2.1, 2.11), the statements to be proven are

$$\mathrm{Re}\,\mathrm{tr}\,\Sigma_B(t) \;=\; \sigma_B \;+\; \sum_{\lambda\in\mathcal{E}_\Delta}\rho'(\lambda)\,\mathrm{Im}\,\mathrm{tr}\,E_{\{\lambda\}}\,[H_B,\Lambda_1]\,\mathrm{e}^{\mathrm{i}H_Bt}\Lambda_2\mathrm{e}^{-\mathrm{i}H_Bt}E_{\{\lambda\}}$$

for Theorem 2 and part of Theorem 1, and

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \operatorname{tr}\Sigma_B(t)\mathrm{d}t \;=\; \sigma_B \tag{2.14}$$

for the other part, where actually the real part of the l.h.s. would suffice. It may be noted that the ρ's allowed by (1.8) form an affine space and that $\Sigma_B(t)$, like $\sigma_E^{(1)}$, $\sigma_E^{(2)}$, is affine in ρ. The relation to σ_B will be made through the following decomposition, which exhibits the same property for this quantity.

Lemma 2. *Let $\Delta\subset\mathbb{R}$ be as in Theorem 1 and let E_-, E_+ be the spectral projections for H_B onto $\{\lambda\,|\,\lambda<\Delta\}$, resp. $\{\lambda\,|\,\lambda>\Delta\}$. Then, for $\lambda_0\in\Delta$*

$$\sigma_B(\lambda_0) \;=\; \mathrm{i}\operatorname{tr}E_-\,[P_{\lambda_0},\Lambda_1]\,\Lambda_2E_- + \mathrm{i}\operatorname{tr}E_+\,[P_{\lambda_0},\Lambda_1]\,\Lambda_2E_+ + \mathrm{i}\operatorname{tr}E_\Delta T_{\lambda_0}E_\Delta\;, \tag{2.15}$$

where $P_{\lambda_0}=E_{(-\infty,\lambda_0)}$,

$$T_{\lambda_0} \;=\; P_{\lambda_0}\Lambda_1P_{\lambda_0}^{\perp}\Lambda_2P_{\lambda_0} - P_{\lambda_0}^{\perp}\Lambda_1P_{\lambda_0}\Lambda_2P_{\lambda_0}^{\perp} \tag{2.16}$$

and the traces are well defined. Moreover, the last term in (2.15) *can be further decomposed as*

$$\mathrm{i}\operatorname{tr}E_\Delta T_{\lambda_0}E_\Delta \;=\; \sum_{\lambda\in\mathcal{E}_\Delta}\mathrm{i}\operatorname{tr}E_{\{\lambda\}}\,[P_{\lambda_0},\Lambda_1]\,\Lambda_2E_{\{\lambda\}}\;, \tag{2.17}$$

with absolutely convergent sum.

Since σ_B is independent of $\lambda_0\in\Delta$, (2.15) with the last term replaced by the r.h.s. of (2.17) also holds if P_{λ_0} is replaced by ρ satisfying (1.8), since $\rho(H_B)=-\int \mathrm{d}\lambda_0\rho'(\lambda_0)P_{\lambda_0}$. The proof of Lemma 2, which is given in Section 3, makes use of

$$1 \;=\; E_- \;+\; E_+ \;+\; E_\Delta\;,\quad E_\Delta=\sum_{\lambda\in\mathcal{E}_\Delta}E_{\{\lambda\}}\;, \tag{2.18}$$

where the sum is strongly convergent. Using this decomposition on $\Sigma_B(t)\in\mathfrak{I}_1$ we obtain

$$\begin{aligned}\operatorname{tr}\Sigma_B(t) \;=\; &\operatorname{tr}E_-\left(\widetilde{\Sigma}'_B+\widetilde{\Sigma}''_B(t)\right)E_- + \operatorname{tr}E_+\left(\widetilde{\Sigma}'_B+\widetilde{\Sigma}''_B(t)\right)E_+\\ &+\operatorname{tr}E_\Delta\Sigma_B(t)E_\Delta\;.\end{aligned} \tag{2.19}$$

Though the two contributions (2.9) to $\Sigma_B(t)$ are not separately trace class, they become so in (2.19). In fact, those of $E_\pm\widetilde{\Sigma}'_BE_\pm$ also appear in (2.15), and $E_\pm\widetilde{\Sigma}''_B(t)E_\pm$ vanish by integration by parts since $E_\pm R(z)$ and $R(z)E_\pm$ are analytic on the support of $\rho(z)$ or of $\rho(z)-1$. We thus find that

$$\operatorname{tr}\Sigma_B(t) \;=\; \sigma_B + \mathrm{i}\int\mathrm{d}\lambda_0\rho'(\lambda_0)\operatorname{tr}E_\Delta T_{\lambda_0}E_\Delta + \operatorname{tr}E_\Delta\Sigma_B(t)E_\Delta\;. \tag{2.20}$$

At this point the analysis of the last term splits into two tracks with the purpose of showing $\sigma_E^{(1)}=\sigma_B$, resp. $\sigma_E^{(2)}=\sigma_B$.

2.1. Track 1. We decompose the projection E_Δ into its atoms as in (2.18), which by

$$X_n \xrightarrow{s} 0\,,\ Y \in \mathfrak{I}_1 \quad \Longrightarrow \quad \|X_n Y\|_1 \to 0\,,\ \|Y X_n^*\|_1 \to 0 \tag{2.21}$$

yields a trace class norm convergent sum for $E_\Delta \Sigma_B(t) E_\Delta$. Thus

$$\operatorname{tr} E_\Delta \Sigma_B(t) E_\Delta \;=\; \sum_{\lambda\in\mathcal{E}_\Delta} \operatorname{tr} E_{\{\lambda\}} \left(\widetilde{\Sigma}'_B + \widetilde{\Sigma}''_B(t)\right) E_{\{\lambda\}}\ .$$

Again, the contributions $E_{\{\lambda\}}\widetilde{\Sigma}'_B E_{\{\lambda\}}$ are themselves trace class as they match those of (2.17), canceling the second term of (2.20). We conclude that

$$\begin{aligned}
&\operatorname{tr}\Sigma_B(t) \\
&\quad= \sigma_B \;+\; \frac{\mathrm{i}}{2\pi}\sum_{\lambda\in\mathcal{E}_\Delta}\int \mathrm{d}m(z)\partial_{\bar z}\rho(z)\operatorname{tr} E_{\{\lambda\}} R(z)\,[H_B,\Lambda_1]\,\Lambda_{2;B}(t) R(z) E_{\{\lambda\}} \\
&\quad= \sigma_B \;-\; \mathrm{i}\sum_{\lambda\in\mathcal{E}_\Delta}\rho'(\lambda)\operatorname{tr} E_{\{\lambda\}}\mathrm{e}^{-\mathrm{i}H_B t}\,[H_B,\Lambda_1]\,\mathrm{e}^{\mathrm{i}H_B t}\Lambda_2 E_{\{\lambda\}}\ ,
\end{aligned} \tag{2.22}$$

where we used that $f(H_B)E_{\{\lambda\}} = f(\lambda)E_{\{\lambda\}}$. By its derivation this sum is absolutely convergent for each t. This proves Thm. 2 and hence $\sigma_E^{(1)} = \sigma_B$.

2.2. Track 2. Here we do not decompose E_Δ, but use (2.10) whose two terms are separately trace class,

$$\operatorname{tr} E_\Delta \Sigma_B(t) E_\Delta \;=\; \operatorname{tr} E_\Delta \Sigma'_B(t) E_\Delta + \operatorname{tr} E_\Delta \Sigma''_B(t) E_\Delta\ .$$

Lemma 3. *For $\Delta \subset \mathbb{R}$ as in Theorem 1 we have*

$$\frac{1}{T}\int_0^T \operatorname{tr} E_\Delta \Sigma''_B(t) E_\Delta\,\mathrm{d}t \;\xrightarrow[T\to\infty]{}\; 0\ , \tag{2.23}$$

and

$$-\mathrm{i}\operatorname{tr} E_\Delta\,[P_{\lambda_0},\Lambda_1]\,(A_{T,B}(\Lambda_2)-\Lambda_2)\,E_\Delta \;\xrightarrow[T\to\infty]{}\; \mathrm{i}\operatorname{tr} E_\Delta T_{\lambda_0} E_\Delta \tag{2.24}$$

for $\lambda_0 \in \Delta$, the expression on the l.h.s. being uniformly bounded in $\lambda_0 \in \Delta$, $T > 0$.

By dominated convergence (2.24) implies

$$\frac{1}{T}\int_0^T \operatorname{tr} E_\Delta \Sigma'_B(t) E_\Delta\,\mathrm{d}t \;\xrightarrow[T\to\infty]{}\; -\mathrm{i}\int \mathrm{d}\lambda_0 \rho'(\lambda_0)\operatorname{tr} E_\Delta T_{\lambda_0} E_\Delta\ .$$

Together with (2.20, 2.23), this proves (2.14) and hence $\sigma_E^{(2)} = \sigma_B$.

2.3. Alternate Track 2. We now show that the last result can also be inferred from (2.22), at least if assumption (1.3) is strengthened to a uniform upper bound on the degeneracies:

$$\dim E_{\{\lambda\}}(H_B) \;\leq\; C_4 < \infty \,, \quad \lambda \in \mathcal{E}_\Delta \,. \tag{2.25}$$

Then, the sum (2.22) is uniformly convergent in $t \in \mathbb{R}$, as stated in

Lemma 4. *Assuming (1.1, 1.2, 2.25), we have*

$$\sum_{\lambda\in\mathcal{E}_\Delta} \sup_{t\in\mathbb{R}} \left|\operatorname{tr} E_{\{\lambda\}} \mathrm{e}^{-\mathrm{i}H_B t} \,[H_B, \Lambda_1]\, \mathrm{e}^{\mathrm{i}H_B t} \Lambda_2 E_{\{\lambda\}}\right| \;<\; \infty \,. \tag{2.26}$$

In order to prove (2.14), it suffices in view of (2.26) to show

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \operatorname{tr} E_{\{\lambda\}} \mathrm{e}^{-\mathrm{i}H_B t} \,[H_B, \Lambda_1]\, \mathrm{e}^{\mathrm{i}H_B t} \Lambda_2 E_{\{\lambda\}} \;=\; 0 \tag{2.27}$$

for each $\lambda \in \mathcal{E}_\Delta$. Because

$$\operatorname{tr} E_{\{\lambda\}} \mathrm{e}^{-\mathrm{i}H_B t} \,[H_B, \Lambda_1]\, \mathrm{e}^{\mathrm{i}H_B t} \Lambda_2 E_{\{\lambda\}} \;=\; \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} \;\operatorname{tr} E_{\{\lambda\}} \mathrm{e}^{-\mathrm{i}H_B t} \Lambda_1 \mathrm{e}^{\mathrm{i}H_B t} \Lambda_2 E_{\{\lambda\}} \,, \tag{2.28}$$

the expression under the limit is just

$$\frac{\mathrm{i}}{T} \left[\operatorname{tr} E_{\{\lambda\}} \mathrm{e}^{-\mathrm{i}H_B t} \Lambda_1 \mathrm{e}^{\mathrm{i}H_B t} \Lambda_2 - \operatorname{tr} E_{\{\lambda\}} \Lambda_1 \Lambda_2\right] \,. \tag{2.29}$$

Since each term inside the square brackets is bounded by $C_4 < \infty$, eq. (2.27) follows. This concludes the alternate proof of $\sigma_E^{(2)} = \sigma_B$.

2.4. Edge conductance in a spectral gap. We conclude this section by showing as mentioned above in the remark following Theorem 1 that

$$-\mathrm{i} \operatorname{tr} \rho'(H_a)\, [H_a, \Lambda_1] \;=\; \sigma_B$$

if $\sigma(H_B) \cap \Delta = \emptyset$. By translation invariance of σ_B, see Lemma 7 below, it suffices to show this for $a = 0$, in which case we drop the subscript a of the edge Hamiltonian. It has been shown in (A.8) of [12] that $\rho'(H)\,[H, \Lambda_1] \in \mathfrak{I}_1$. Since $\Lambda_{2,a} := \Lambda_2(\cdot - a) \xrightarrow{s} 1$ as $a \to \infty$, we have by (2.21)

$$\begin{aligned} -\,\mathrm{i} \operatorname{tr} \rho'(H)\, [H, \Lambda_1] \;&=\; -\mathrm{i} \lim_{a\to\infty} \operatorname{tr} \rho'(H)\, [H, \Lambda_1]\, \Lambda_{2,a} \\ &=\; -\mathrm{i} \lim_{a\to\infty} \operatorname{tr} \rho'(H^a)\, [H^a, \Lambda_1]\, \Lambda_2 \,. \end{aligned} \tag{2.30}$$

Here H^a is the operator on $\ell^2(\mathbb{Z} \times \mathbb{Z}_a)$ obtained from H by a shift $(0, -a)$; it is not the restriction to $\mathbb{Z} \times \mathbb{Z}_a$ of a fixed Bulk Hamiltonian H_B, as H_a was, but instead of an equally shifted one, H_B^a. The estimates (1.1, 1.6) therefore still apply, which is all that matters for (2.12, 2.13). The r.h.s. of (2.30) thus equals $\lim_{a\to\infty} \operatorname{tr} \Sigma_B^a(0)$, where $\Sigma_B^a(t)$ pertains to H_B^a. Since the sum in (2.22) vanishes, $\operatorname{tr} \Sigma_B^a(t) = \sigma_B^a$, which is independent of a.

3. Details of the Proof

We give some details about the Helffer-Sjöstrand representations (2.6). The integral is over $z = x + \mathrm{i}y \in \mathbb{C}$ with measure $\mathrm{d}m(z) = \mathrm{d}x\mathrm{d}y$, $\partial_{\bar z} = \partial_x + \mathrm{i}\partial_y$, and $\rho(z)$ is a quasi-analytic extension of $\rho(x)$ which, see [18], for given n can be chosen so that

$$\int \mathrm{d}m(z)\, |\partial_{\bar z}\rho(z)|\, |y|^{-p-1} \ \le\ C \sum_{k=0}^{n+2} \left\|\rho^{(k)}\right\|_{k-p-1} \tag{3.1}$$

for $p = 1, ..., n$, provided the appearing norms $\|f\|_k = \int \mathrm{d}x(1+x^2)^{\frac{k}{2}}|f(x)|$ are finite. This is the case for ρ with $\rho' \in C_0^\infty(\mathbb{R})$. For $p = 1$ this shows that (2.6b) is norm convergent. The integral (2.6a), which would correspond to the case $p = 0$, is nevertheless a strongly convergent improper integral, see e.g., (A.12) of [12].

A further preliminary is the Combes-Thomas bound [8]

$$\left\|\mathrm{e}^{\delta\ell(x)} R_a(z) \mathrm{e}^{-\delta\ell(x)}\right\| \ \le\ \frac{C}{|\operatorname{Im} z|}\ , \tag{3.2}$$

where δ can be chosen as

$$\delta^{-1} \ =\ C\left(1 + |\operatorname{Im} z|^{-1}\right) \tag{3.3}$$

for some (large) $C > 0$ and $\ell(x)$ is any Lipschitz function on $\mathbb{Z}^2$ with

$$|\ell(x) - \ell(y)| \ \le\ |x-y| \tag{3.4}$$

(see e.g. [2, Appendix D] for details).

Lemma 5. *We have*

$$[H_a, \Lambda_1]\, \Lambda_{2;a}(t) \ \in\ \mathfrak{I}_1\ , \tag{3.5}$$

and for $\rho \in C^\infty(\mathbb{R})$ with $\operatorname{supp}\rho'$ compact also

$$[\rho(H_a), \Lambda_1]\, \Lambda_{2;a}(t) \ \in\ \mathfrak{I}_1\ . \tag{3.6}$$

In particular, $Z(a,t)$ as given in (2.8) is trace class.

Proof. We first prove the finite propagation speed estimate (see [13] and [23]):

Let $\mu > 0$ be as in (1.1). Then, for $0 \le \delta \le \mu$ and ℓ as (3.4),

$$\left\|\mathrm{e}^{\delta\ell(x)}\mathrm{e}^{\mathrm{i}H_a t}\mathrm{e}^{-\delta\ell(x)}\right\| \ \le\ \mathrm{e}^{C|t|} \tag{3.7}$$

for some $C < \infty$.

Indeed, let $A(t) = \mathrm{e}^{\delta\ell(x)}\mathrm{e}^{\mathrm{i}H_a t}\mathrm{e}^{-\delta\ell(x)}$.

$$\frac{\mathrm{d}}{\mathrm{d}t}A(t)^*A(t) \ =\ \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{-\delta\ell(x)}\mathrm{e}^{-\mathrm{i}H_a t}\mathrm{e}^{2\delta\ell(x)}\mathrm{e}^{\mathrm{i}H_a t}\mathrm{e}^{-\delta\ell(x)} \ =\ A(t)^*BA(t)\ ,$$

where $B = -\mathrm{i}\mathrm{e}^{-\delta\ell(x)}\left[H_a, \mathrm{e}^{2\delta\ell(x)}\right]\mathrm{e}^{-2\delta\ell(x)}$ has matrix elements

$$\mathrm{i}B(x,x') \ =\ H_a(x,x')(\mathrm{e}^{\delta(\ell(x')-\ell(x))} - \mathrm{e}^{\delta(\ell(x)-\ell(x'))}).$$

By (1.1) which, as remarked in the Introduction, is inherited by H_a, and by Holmgren's bound

$$\|B\| \;\le\; \max\left(\sup_x \sum_{x'} |B(x,x')|\,,\; \sup_{x'} \sum_{x} |B(x,x')|\right)\,, \tag{3.8}$$

we have $2C := \|B\| < \infty$ and hence

$$\|A(t)\|^2 \;=\; \|A(t)^*A(t)\| \;\le\; \mathrm{e}^{2C|t|}\,.$$

We factorize

$$[H_a, \Lambda_1]\,\Lambda_{2;a}(t) \;=\; [H_a, \Lambda_1]\,\mathrm{e}^{\delta|x_1|} \cdot \mathrm{e}^{-\delta|x_1|}\mathrm{e}^{-\delta|x_2|} \cdot \mathrm{e}^{\delta|x_2|}\Lambda_{2;a}(t)\,,$$

and note that

$$\left\|\mathrm{e}^{-\delta|x_1|}\mathrm{e}^{-\delta|x_2|}\right\|_1 \;\le\; C\delta^{-2}\,, \tag{3.9}$$

since this is a summable function of $(x_1, x_2) \in \mathbb{Z}^2$. It is therefore enough for (3.5) to show

$$\left\|[H_a, \Lambda_1]\,\mathrm{e}^{\delta|x_1|}\right\| \;\le\; C\,, \tag{3.10}$$

$$\left\|\mathrm{e}^{\delta|x_2|}\Lambda_{2;a}(t)\right\| \;\le\; C\mathrm{e}^{\delta a} \tag{3.11}$$

for small δ, where the first estimate also holds for $a = B$. Indeed, the first operator has matrix elements

$$T(x,x') \;=\; H_a(x,x')(\Lambda_1(x) - \Lambda_1(x'))\mathrm{e}^{\delta|x_1'|}\,.$$

They vanish if $|x_1 - x_1'| \le |x_1'|$ since $x_1' \ge 0$ (resp. $x_1' < 0$) then implies the same for x_1. Therefore

$$|T(x,x')| \;\le\; 2\,|H_a(x,x')|\,\mathrm{e}^{\delta|x_1 - x_1'|} \;\le\; 2\,|H_a(x,x')|\,\mathrm{e}^{\delta|x-x'|}\,,$$

which together with $T(x,x) = 0$ yields $|T(x,x')| \le C|H_a(x,x')|(\mathrm{e}^{\delta|x-x'|} - 1)$. Now (3.10) follows from (1.1) and (3.8). The estimate for

$$\mathrm{e}^{\delta|x_2|}\Lambda_{2;a}(t) = \mathrm{e}^{\delta|x_2|}\mathrm{e}^{\mathrm{i}H_a t}\mathrm{e}^{-\delta|x_2|} \cdot \mathrm{e}^{\delta|x_2|}\Lambda_2\mathrm{e}^{-\mathrm{i}H_a t}$$

follows from (3.7) and from $\|\mathrm{e}^{\delta|x_2|}\Lambda_2\| = \mathrm{e}^{\delta a} < \infty$.

The proof of (3.6) is similar: Using (2.6) we write

$$[\rho(H_a), \Lambda_1] \;=\; \frac{1}{2\pi}\int \mathrm{d}m(z)\partial_{\bar z}\rho(z)\,[R_a(z), \Lambda_1] \tag{3.12}$$

and claim that

$$\left\|[R_a(z), \Lambda_1]\,\mathrm{e}^{\delta|x_1|}\right\| \;\le\; \frac{C}{|\operatorname{Im} z|^2} \tag{3.13}$$

for $\delta = \delta(z)$ as in (3.3). Together with (3.1, 3.9, 3.11) this implies (3.6). To derive (3.13), note that the operator to be bounded is $-R_a(z)[H_a\,,\ \Lambda_1]\mathrm{e}^{\delta|x_1|} \cdot \mathrm{e}^{-\delta|x_1|}R_a(z)\mathrm{e}^{\delta|x_1|}$ and the bound follows from (3.2, 3.10).

The conclusion about $Z(a,t)$ follows from (3.6) at $t = 0$ and (3.1, 3.5). $\square$

3.1. Proof of Lemma 1. It follows from (3.6) that $\Sigma'_a(t)$ is trace class. While (3.13) holds uniformly in a, including the Bulk case, (3.11) fails in this respect. Nevertheless $\Sigma'_B(t) \in \mathfrak{I}_1$, since

$$\sup_{a,B} \left\| \mathrm{e}^{\delta|x_2|}(\Lambda_{2;a}(t) - \Lambda_2) \right\| \;\leq\; C \tag{3.14}$$

for t in a compact interval. In fact

$$\mathrm{e}^{\delta|x_2|}\,(\Lambda_{2;a}(t) - \Lambda_2) \;=\; \mathrm{e}^{\delta|x_2|} \int_0^t \mathrm{e}^{\mathrm{i}H_a s}\mathrm{i}\,[H_a, \Lambda_2]\,\mathrm{e}^{-\mathrm{i}H_a s}$$

with

$$\sup_{a,B} \left\| \mathrm{e}^{\delta|x_2|}\mathrm{e}^{\mathrm{i}H_a t}\,[H_a, \Lambda_2]\,\mathrm{e}^{-\mathrm{i}H_a t} \right\| \;\leq\; C\;, \tag{3.15}$$

because of (3.7) and of $\left\|\mathrm{e}^{\delta|x_2|}\,[H_a, \Lambda_2]\right\| \leq C$, c.f. (3.10).

To prove (2.12) we use (3.12) and $\mathcal{J}_a^*\mathcal{J}_a = 1$ to write

$$\begin{aligned}\mathcal{J}_a\Sigma'_a(t)\mathcal{J}_a^* \;&=\; -\frac{1}{2\pi}\int \mathrm{d}m(z)\partial_{\bar z}\rho(z)\\ &\times \mathcal{J}_a\,[R_a(z), \Lambda_1]\,\mathrm{e}^{\delta|x_1|}\mathcal{J}_a^* \cdot \mathrm{e}^{-\delta|x_1|}\mathrm{e}^{-\delta|x_2|} \cdot \mathcal{J}_a\mathrm{e}^{\delta|x_2|}\,(\Lambda_{2;a}(t) - \Lambda_2)\,\mathcal{J}_a^*\end{aligned}$$

It is enough to establish convergence to the bulk expression pointwise in z, since domination is provided by (3.13, 3.9, 3.14, 3.1). We thus may show

$$\mathcal{J}_a\,[R_a(z), \Lambda_1]\,\mathrm{e}^{\delta|x_1|}\mathcal{J}_a^* \;\xrightarrow[a\to\infty]{s}\; [R_B(z), \Lambda_1]\,\mathrm{e}^{\delta|x_1|}\;, \tag{3.16}$$

$$\mathcal{J}_a\,(\Lambda_{2;a}(t) - \Lambda_2)\,\mathcal{J}_a^*\mathrm{e}^{\delta|x_2|} \;\xrightarrow[a\to\infty]{s}\; (\Lambda_{2B}(t) - \Lambda_2)\,\mathrm{e}^{\delta|x_2|}\;. \tag{3.17}$$

Since the l.h.s.'s are uniformly bounded in a by (3.13, 3.14) it suffices to prove convergence on the dense subspace of compactly supported states in $\ell^2(\mathbb{Z}\times\mathbb{Z})$, which amounts to dropping $\mathrm{e}^{\delta|x_i|}$ in (3.16, 3.17). Eq. (1.5) implies the geometric resolvent identity $\mathcal{J}_aR_a(z) - R_B(z)\mathcal{J}_a = -R_B(z)E_aR_a(z)$, and by taking the adjoint

$$\mathcal{J}_aR_a(z)\mathcal{J}_a^* - R_B(z) \;=\; -\left(\mathcal{J}_aR_a(z)E_a^* + 1 - \mathcal{J}_a\mathcal{J}_a^*\right)R_B(z) \;\xrightarrow[a\to\infty]{s}\; 0$$

because $E_a^* \xrightarrow[a\to\infty]{s} 0$ by (1.6) and because $1 - \mathcal{J}_a\mathcal{J}_a^* \xrightarrow[a\to\infty]{s} 0$ is the projection onto states supported in $\{x_2 < -a\}$. This implies [30, Thm. VIII.20]

$$\operatorname*{s-lim}_{a\to\infty} \mathcal{J}_af(H_a)\mathcal{J}_a^* \;=\; f(H_B) \tag{3.18}$$

for any bounded continuous function f, and in particular the modified limits (3.16, 3.17). The proof of (2.13) is similar. We write the integrand of $\mathcal{J}_a\Sigma''_a(t)\mathcal{J}_a^*$ as

$$\mathcal{J}_a\,[R_a(z), \Lambda_1]\,\mathrm{e}^{\delta|x_1|}\mathcal{J}_a^* \cdot \mathrm{e}^{-\delta|x_1|}\mathrm{e}^{-\delta|x_2|} \cdot \mathcal{J}_a\mathrm{e}^{\delta|x_2|}\,[H_a, \Lambda_{2;a}(t)]\,R_a(z)\mathcal{J}_a^*\;.$$

Since the estimates for the first two factors have already been given, all we need are

$$\sup_{a,B}\left\|\mathrm{e}^{\delta|x_2|}\left[H_a,\Lambda_{2;a}(t)\right]\right\| \;\le\; C\;,$$

$$\mathcal{J}_a\left[H_a,\Lambda_{2;a}(t)\right]\mathcal{J}_a^* \;\xrightarrow[a\to\infty]{s}\; \left[H_B,\Lambda_{2B}(t)\right]\;.$$

The first estimate is just (3.15) and the second is again implied by (3.18). □

3.2. Proof of Lemma 2. Let $P_{\lambda_0}^{\perp}=1-P_{\lambda_0}$. By the definition (1.4) we have

$$\sigma_B(\lambda_0) \;=\; \mathrm{i}\,\mathrm{tr}\left(P_{\lambda_0}\Lambda_1P_{\lambda_0}^{\perp}\Lambda_2P_{\lambda_0}-P_{\lambda_0}\Lambda_2P_{\lambda_0}^{\perp}\Lambda_1P_{\lambda_0}\right)\;.$$

Since the two terms are separately trace class by (A.2), we also have $-\mathrm{i}\sigma_B(\lambda_0)=\mathrm{tr}\,T_{\lambda_0}$ with T_{λ_0} as in (2.16); see (2.2). Now (2.18) yields

$$-\mathrm{i}\sigma_B(\lambda_0) \;=\; \mathrm{tr}\left(E_-T_{\lambda_0}E_- \;+\; E_+T_{\lambda_0}E_+ \;+\; \sum_{\lambda\in\mathcal{E}_\Delta}E_{\{\lambda\}}T_{\lambda_0}E_{\{\lambda\}}\right)\;,$$

and the claim follows from

$$\mathrm{tr}\,PT_{\lambda_0}P=\mathrm{tr}\,P\left[P_{\lambda_0},\Lambda_1\right]\Lambda_2P$$

for $P=P^*$ with $PP_{\lambda_0}^{\perp}=0$ or $PP_{\lambda_0}=0$, since one or the other holds true for $P=E_\pm,E_{\{\lambda\}}$. Indeed, in the first case, which also entails $P_{\lambda_0}^{\perp}P=0$, we have

$$\begin{aligned}PT_{\lambda_0}P \;&=\; PP_{\lambda_0}\Lambda_1P_{\lambda_0}^{\perp}\Lambda_2P_{\lambda_0}P \;=\; P\left(P_{\lambda_0}\Lambda_1\Lambda_2-\Lambda_1P_{\lambda_0}\Lambda_2\right)P\\ &=\; P\left[P_{\lambda_0},\Lambda_1\right]\Lambda_2P\;.\end{aligned}$$

The other case is similar:

$$\begin{aligned}PT_{\lambda_0}P \;&=\; -PP_{\lambda_0}^{\perp}\Lambda_1P_{\lambda_0}\Lambda_2P_{\lambda_0}^{\perp}P \;=\; -P\left(P_{\lambda_0}^{\perp}\Lambda_1\Lambda_2-\Lambda_1P_{\lambda_0}^{\perp}\Lambda_2\right)P\\ &=\; -P\left[P_{\lambda_0}^{\perp},\Lambda_1\right]\Lambda_2P \;=\; P\left[P_{\lambda_0},\Lambda_1\right]\Lambda_2P\;.\end{aligned}$$ □

3.3. Consequences of localization. We now discuss the technical consequences of assumption (1.2). In fact, all that we say in this section is a consequence of the following (weaker) estimate

$$\sup_{g\in B_1(\Delta)}\sum_{x,x'\in\mathbb{Z}^2}\left|g(H_B)(x,x')\right|\mathrm{e}^{-\varepsilon|x|}\mathrm{e}^{\mu|x-x'|} \;=:\; D_\varepsilon \;<\; \infty\;, \tag{3.19}$$

for every $\varepsilon>0$, where the factor $(1+|x|)^{-\nu}$ of (1.2) has been replaced by an exponential. Note that (3.19) follows from (1.2) since $\mathrm{e}^{-\varepsilon|x|}\le C_{\varepsilon,\nu}(1+|x|)^{-\nu}$. (We require (1.2) to prove integrality of $2\pi\sigma_B$ (Prop. 3 below), otherwise (3.19) would suffice for the results described here.)

In terms of operators, rather than of matrix elements, (3.19) implies that for some $\mu>0$ and all $\varepsilon>0$

$$\sup_{g,\ell}\left\|\mathrm{e}^{\mu\ell(x)}\mathrm{e}^{-\varepsilon|x|}g(H_B)\mathrm{e}^{-\mu\ell(x)}\right\| \;\le\; D_\varepsilon \;<\; \infty\;, \tag{3.20}$$

where the supremum with $g \in B_1(\Delta)$ is also taken over Lipschitz functions ℓ as in (3.4). In fact, the norm in (3.20) is estimated by Holmgren's bound (3.8) as the larger of

$$\sup_x \sum_{x'} \mathrm{e}^{\mu(\ell(x)-\ell(x'))} \mathrm{e}^{-\varepsilon|x|} \, |g(H_B)(x,x')| \tag{3.21}$$

and a similar quantity with x, x' under the supremum and summation interchanged. After bounding the supremum by a sum, both quantities are estimated by (3.19). Conversely, we take $\ell(x) = |x - x'|$ and consider the (x,x') matrix element of the operator in (3.20),

$$\mathrm{e}^{\mu|x-x'|} \mathrm{e}^{-\varepsilon|x|} \, |g(H_B)(x,x')| \;\leq\; D_\varepsilon \; . \tag{3.22}$$

The sum in (3.19) is finite if μ is replaced there by $\mu/2$ and ε by 2ε.

We say that a bounded operator X is *confined in direction* i ($i = 1, 2$) if for some $\delta > 0$ and all (small) $\varepsilon > 0$

$$\|X\|^{(i)}_{\varepsilon,\delta} \;:=\; \left\| X \mathrm{e}^{-\varepsilon|x|} \mathrm{e}^{\delta|x_i|} \right\| \;<\; \infty \; . \tag{3.23}$$

Bounds of a similar form are (3.13, 3.14), where a weight was applied to an operator X, which could have as well been replaced by X^*. Equivalently, the weight could have been placed on either side of X. Here, by contrast, dynamical localization will allow to establish (3.23) for some operators X, but not for their adjoints. The asymmetry originates from the following: if X is confined, so are BX for B bounded and $Xg(H_B)$ for $g \in B_1(\Delta)$, with

$$\|BX\|^{(i)}_{\varepsilon,\delta} \;\leq\; \|B\| \; \|X\|^{(i)}_{\varepsilon,\delta} \;, \tag{3.24}$$

$$\|Xg(H_B)\|^{(i)}_{\varepsilon,\delta} \leq\; D_{\frac{\varepsilon}{2}} \, \|X\|^{(i)}_{\frac{\varepsilon}{2},\delta} \tag{3.25}$$

for small $\delta > 0$. In fact,

$$\begin{aligned} &\left\| Xg(H_B) \mathrm{e}^{-\varepsilon|x|} \mathrm{e}^{\delta|x_2|} \right\| \\ &\qquad \leq\; \left\| X \mathrm{e}^{-\frac{\varepsilon}{2}|x|} \mathrm{e}^{\delta|x_2|} \right\| \cdot \left\| \mathrm{e}^{-(\delta|x_2| - \frac{\varepsilon}{2}|x|)} g(H_B) \mathrm{e}^{-\frac{\varepsilon}{2}|x|} \mathrm{e}^{(\delta|x_2| - \frac{\varepsilon}{2}|x|)} \right\| \;, \end{aligned}$$

and for sufficiently small $\varepsilon, \delta > 0$ the Lipschitz norm of $\delta|x_2| - \frac{\varepsilon}{2}|x|$ is smaller than μ, whence (3.20) applies.

Lemma 6. *Let $S \subset \mathbb{R}$ be a Borel set that either contains or is disjoint from $\{\lambda | \lambda < \Delta\}$ and similarly for $\{\lambda | \lambda > \Delta\}$, i.e., $E_S \in B_1(\Delta)$. Let X be a confined operator in direction i ($i = 1, 2$).*

i) The following operators are also confined in direction i, as indicated by the estimates

$$\|[X, g(H_B)]\|^{(i)}_{\varepsilon,\delta} \;\leq\; C \, \|X\|^{(i)}_{\frac{\varepsilon}{2},\delta} \;, \quad (g \in B_1(\Delta)) \;, \tag{3.26}$$

$$\left\| E_S^\perp X E_S \right\|^{(i)}_{\varepsilon,\delta} \;\leq\; C \, \|X\|^{(i)}_{\frac{\varepsilon}{2},\delta} \; . \tag{3.27}$$

ii) If in addition $S \subset \Delta$, then the following operators are also confined

$$\|[H_B, A_{T,B}(X)]\, E_S\|^{(i)}_{\varepsilon,\delta} \;\le\; \frac{C}{T}\,\|X\|^{(i)}_{\frac{\varepsilon}{2},\delta}\;, \tag{3.28}$$

$$\|(A_{T,B}(X) - X)\, E_S\|^{(i)}_{\varepsilon,\delta} \;\le\; C\,\|X\|^{(i)}_{\frac{\varepsilon}{2},\delta}\;, \tag{3.29}$$

and given $S' \subset \mathbb{R}$ with $d = \mathrm{dist}(S, S') > 0$,

$$\|E_{S'} A_{T,B}(X) E_S\|^{(i)}_{\varepsilon,\delta} \;\le\; \frac{C}{T}\,\|X\|^{(i)}_{\frac{\varepsilon}{2},\delta}\;. \tag{3.30}$$

iii) Properties (i, ii) also hold for $X = \Lambda_i$, with $\|X\|^{(i)}_{\frac{\varepsilon}{2},\delta}$ replaced by 1.

The constants C depend on ε, δ, but not on the remaining quantities, except for (3.30) *which depends on d.*

The main use of confined operators will be through the following remark: If X_i, $(i = 1, 2)$, is confined in direction i, then $X_2 X_1^* \in \mathfrak{I}_1$ with

$$\|X_2 X_1^*\|_1 \;\le\; C\,\|X_2\|^{(2)}_{\varepsilon,\delta}\,\|X_1\|^{(1)}_{\varepsilon,\delta} \tag{3.31}$$

for $2\varepsilon < \delta$. In particular, if also $X_1^* X_2 \in \mathfrak{I}_1$, (3.31) is a bound for $\mathrm{tr}\, X_1^* X_2 = \mathrm{tr}\, X_2 X_1^*$. Indeed, (3.31) follows from $\mathrm{e}^{-\delta|x_2|}\mathrm{e}^{2\varepsilon|x|}\mathrm{e}^{-\delta|x_1|} = \mathrm{e}^{-(\delta-2\varepsilon)|x|} \in \mathfrak{I}_1$.

3.4. Proof of Lemma 6. For X confined, (3.26) is implied by (3.24, 3.25). We thus consider $X = \Lambda_i$, where it is enough to estimate

$$\begin{aligned}[\Lambda_i, g(H_B)]\,\mathrm{e}^{-\varepsilon|x|}\mathrm{e}^{\pm\delta x_i} \;&=\; \Lambda_i g(H_B)(1-\Lambda_i)\mathrm{e}^{-\varepsilon|x|}\mathrm{e}^{\pm\delta x_i}\\ &\qquad + (1-\Lambda_i) g(H_B)\Lambda_i \mathrm{e}^{-\varepsilon|x|}\mathrm{e}^{\pm\delta x_i}\;.\end{aligned}$$

In the + case, for instance, the second term is bounded because $\Lambda_i \mathrm{e}^{\delta x_i}$ is. By (3.20) this holds for the first one too.

From now on the switch functions and the confined operators will be treated simultaneously. Eq. (3.27) follows from (3.26) and $E_S^\perp X E_S = E_S^\perp [X, E_S]$. To prove (3.28) we consider

$$\begin{aligned}T\cdot \mathrm{i}\,[H_B, A_{T,B}(X)]\,E_S \;&=\; (\mathrm{e}^{\mathrm{i}H_B T} X \mathrm{e}^{-\mathrm{i}H_B T} - X)E_S\\ &=\; \mathrm{e}^{\mathrm{i}H_B T}\left(X\mathrm{e}^{-\mathrm{i}H_B T}E_S - \mathrm{e}^{-\mathrm{i}H_B T}E_S X\right)E_S \;-\; E_S^\perp X E_S\;.\end{aligned} \tag{3.32}$$

The term in parentheses is bounded by (3.26) for $g(\lambda) = \mathrm{e}^{-\mathrm{i}\lambda T}E_S(\lambda)$. The norm (3.23) of (3.32) is uniformly bounded in $T \in \mathbb{R}$ by (3.24, 3.25, 3.27). The same bound applies to

$$(A_{T,B}(X) - X)E_S \;=\; \frac{1}{T}\int_0^T \mathrm{d}t(\mathrm{e}^{\mathrm{i}H_B t}X\mathrm{e}^{-\mathrm{i}H_B t} - X)E_S\;.$$

We now turn to (3.30), which is related to an integration by parts lemma of [19]. Since $S \subset \Delta$ and $d > 0$, there is a contour γ in the complex plane (of length $\le 4|\Delta| + 2d$) encircling S once, but not S', at a distance $\ge d/2$ from both. Then

$$\widetilde{X} \;=\; \frac{1}{2\pi}\int_\gamma \mathrm{d}z R(z)E_{S'}XE_S R(z)$$

is convergent in the norm (3.23) because of (3.24, 3.25, 3.27) (note that $(2/d)\cdot E_S(\lambda)(z-\lambda)^{-1}\in B_1(\varDelta)$). Its commutator with H_B is

$$\begin{aligned}\mathrm{i}\left[H_B,\widetilde{X}\right] &= -\frac{1}{2\pi\mathrm{i}}\int_\gamma \mathrm{d}z\,[H_B-z,R(z)E_{S'}XE_SR(z)]\\ &= -\frac{1}{2\pi\mathrm{i}}\int_\gamma \mathrm{d}z(E_{S'}XE_SR(z)-R(z)E_{S'}XE_S) \;=\; E_{S'}XE_S\ .\end{aligned}$$

Therefore, $E_{S'}A_{T,B}(X)E_S \;=\; A_{T,B}(E_{S'}XE_S) \;=\; E_{S'}\mathrm{i}\left[H_B,A_{T,B}(\widetilde{X})\right]E_S$ and the claim follows from (3.28). □

3.5. Proof of Lemma 3. We first prove (2.23) and begin by recalling, see (2.10, 1.14), that

$$\frac{1}{T}\int_0^T \operatorname{tr}E_\varDelta\varSigma_B''(t)E_\varDelta \;=\; \frac{\mathrm{i}}{2\pi}\int \mathrm{d}m(z)\partial_{\bar z}\rho(z)\operatorname{tr}E_\varDelta R(z)\left[H_B,\varLambda_1\right]\cdot \\ \cdot R(z)\left[H_B,A_{T,B}(\varLambda_2)\right]R(z)E_\varDelta\ . \quad (3.33)$$

By (3.24, 3.25, 3.28) we have for small $\delta>0$

$$\|R(z)\left[H_B,A_{T,B}(\varLambda_2)\right]R(z)E_\varDelta\|^{(2)}_{\varepsilon,\delta} \;\le\; \frac{C}{T}|\operatorname{Im}z|^{-2}\ ,$$

and, together with (3.10),

$$\|\left[H_B,\varLambda_1\right]R(z)E_\varDelta\|^{(1)}_{\varepsilon,\delta} \;\le\; C|\operatorname{Im}z|^{-1}\ .$$

By (3.31) the trace in (3.33) is bounded by a constant times $T^{-1}|\operatorname{Im}z|^{-3}$. As the constant is independent of z, (2.23) now follows by means of (3.1).

The operator under the trace in (2.24) is

$$\begin{aligned}&E_\varDelta P_{\lambda_0}\varLambda_1(A_{T,B}(\varLambda_2)-\varLambda_2)E_\varDelta - E_\varDelta\varLambda_1P_{\lambda_0}(A_{T,B}(\varLambda_2)-\varLambda_2)E_\varDelta\\ &\qquad= E_\varDelta P_{\lambda_0}\varLambda_1P_{\lambda_0}^\perp\cdot(A_{T,B}(\varLambda_2)-\varLambda_2)E_\varDelta\\ &\qquad\quad - E_\varDelta P_{\lambda_0}^\perp\varLambda_1P_{\lambda_0}\cdot(A_{T,B}(\varLambda_2)-\varLambda_2)E_\varDelta\ . \end{aligned}\quad (3.34)$$

We claim that the two terms on the r.h.s. are separately trace class. In fact (3.27) implies $\|P_{\lambda_0}\varLambda_1P_{\lambda_0}^\perp\mathrm{e}^{-\varepsilon|x|}\mathrm{e}^{\delta|x_1|}\|\le C$, and similarly with $P_{\lambda_0},P_{\lambda_0}^\perp$ interchanged, and the bound (3.14) also applies with $A_{T,B}(\varLambda_2)$ in place of $\varLambda_{2,B}(t)$. (Note however that the bound so obtained is not uniform in T.)

A factor P_{λ_0}, resp. $P_{\lambda_0}^\perp$, may now be cycled around the traces of the two terms on the r.h.s. of (3.34). The trace (2.24) thus equals

$$\begin{aligned}&\operatorname{tr}E_\varDelta P_{\lambda_0}\varLambda_1P_{\lambda_0}^\perp\cdot P_{\lambda_0}^\perp A_{T,B}(\varLambda_2)P_{\lambda_0}E_\varDelta\\ &\qquad-\operatorname{tr}E_\varDelta P_{\lambda_0}^\perp\varLambda_1P_{\lambda_0}\cdot P_{\lambda_0}A_{T,B}(\varLambda_2)P_{\lambda_0}^\perp E_\varDelta - \operatorname{tr}E_\varDelta T_{\lambda_0}E_\varDelta\ ,\end{aligned}\quad (3.35)$$

where we used that the two terms of T_{λ_0}, see (2.16), are separately trace class.

We next show that the first two terms of (3.35) are uniformly bounded in $\lambda_0 \in \Delta$, $T > 0$. Indeed, $X_1 = P_{\lambda_0}^{\perp} \Lambda_1 P_{\lambda_0} E_\Delta$ and $X_2 = P_{\lambda_0}^{\perp} A_{T,B}(\Lambda_2) P_{\lambda_0} E_\Delta = P_{\lambda_0}^{\perp}(A_{T,B}(\Lambda_2) - \Lambda_2) P_{\lambda_0} E_\Delta + P_{\lambda_0}^{\perp} \Lambda_2 P_{\lambda_0} E_\Delta$ are uniformly confined by (3.27, 3.29) and the conclusion is by (3.31).

Finally, we will show that these two terms vanish as $T \to \infty$, pointwise in $\lambda_0 \in \Delta$. The first one is split according to $P_{\lambda_0} = P_\lambda + (P_{\lambda_0} - P_\lambda)$ for any $\lambda < \lambda_0$, $\lambda \in \Delta$:

$$\begin{aligned}
&\operatorname{tr} P_{\lambda_0}^{\perp} A_{T,B}(\Lambda_2) P_{\lambda_0} E_\Delta \cdot E_\Delta P_{\lambda_0} \Lambda_1 P_{\lambda_0}^{\perp} \\
&\qquad = \operatorname{tr} P_{\lambda_0}^{\perp} A_{T,B}(\Lambda_2) P_{\lambda} E_\Delta \cdot E_\Delta P_{\lambda_0} \Lambda_1 P_{\lambda_0}^{\perp} \\
&\qquad\quad + \operatorname{tr} P_{\lambda_0}^{\perp} A_{T,B}(\Lambda_2) P_{\lambda_0} E_\Delta \cdot (P_{\lambda_0} - P_\lambda) \cdot E_\Delta P_{\lambda_0} \Lambda_1 P_{\lambda_0}^{\perp} \\
&\qquad \equiv \mathrm{I} \ + \ \mathrm{II}\ .
\end{aligned}$$

In II, we extract the weights of the confined operators, so that the middle factor becomes

$$\begin{aligned}
&\mathrm{e}^{2\varepsilon|x|}\mathrm{e}^{-\frac{\delta}{2}(|x_1|+|x_2|)} \cdot \mathrm{e}^{-\varepsilon|x|}\mathrm{e}^{\frac{\delta}{2}(|x_1|-|x_2|)}(P_{\lambda_0} - P_\lambda)\mathrm{e}^{\frac{\delta}{2}(|x_2|-|x_1|)}\mathrm{e}^{-\varepsilon|x|} \cdot \\
&\qquad\qquad\qquad\qquad\qquad\qquad \cdot \mathrm{e}^{-\frac{\delta}{2}(|x_1|+|x_2|)}\mathrm{e}^{2\varepsilon|x|}\ .
\end{aligned}$$

For $\delta/2 > 2\varepsilon$ the operators on the sides are trace class, and the middle one is uniformly bounded in $\lambda \in \Delta$ by (3.20). Moreover, it converges weakly to zero as $\lambda \uparrow \lambda_0$, as this holds true by $P_{\lambda_0} - P_\lambda \xrightarrow{s} 0$ for matrix elements between states from the dense subspace of compactly supported states in $\ell^2(\mathbb{Z}^2)$. Using

$$X_n \xrightarrow{w} 0\ , \quad Y_1\ ,\ Y_2 \in \mathfrak{I}_1 \quad \Longrightarrow \quad \|Y_1 X_n Y_2\|_1 \to 0\ ,$$

we conclude that II can be made uniformly small in T by picking λ close to λ_0. The term I is then seen to be $\mathcal{O}(T^{-1})$ by (3.30) with $S = (-\infty, \lambda) \cap \Delta$ and $S' = [\lambda_0, \infty)$.

The second trace in (3.35) is dealt with slightly differently. We insert $P_{\lambda_0} = P_\lambda + E_\Delta(P_{\lambda_0} - P_\lambda)E_\Delta$ for $\lambda < \lambda_0$, $\lambda \in \Delta$, which yields two well-defined traces. The second can be made uniformly small in T, as was the case for II above. The first one, which by (2.2) equals $\operatorname{tr} P_\lambda A_{T,B}(\Lambda_2) P_{\lambda_0}^{\perp} E_\Delta \cdot E_\Delta P_{\lambda_0}^{\perp} \Lambda_1 P_\lambda$, is $\mathcal{O}(T^{-1})$ by (3.30), this time with $S = [\lambda_0, \infty) \cap \Delta$, $S' = (-\infty, \lambda)$. □

3.6. Proof of Lemma 4. We shall need a particular choice of basis $\{\psi_{\lambda;j}\}$ for $\operatorname{ran} E_{\{\lambda\}}$, which is related to a SULE basis [11]. (The issue is only of relevance if $\lambda \in \mathcal{E}_\Delta$ is degenerate, since otherwise ψ_λ is unique up to a phase.) We claim a basis can be chosen so that (3.20) applies not only to $g(H_\lambda) = E_{\{\lambda\}} = \sum \psi_{\lambda;j}\,(\psi_{\lambda;j}\ ,\ \cdot\ )$, but also to the rank one projections into which it is decomposed (upon changing μ, D_ε, depending on C_4). Since $\|\phi\,(\psi\ \cdot\ )\| = \|\phi\|\,\|\psi\|$, this amounts to

$$\sup_{\ell} \left\| \mathrm{e}^{\mu\ell(x)}\mathrm{e}^{-\varepsilon|x|}\psi_{\lambda;j} \right\| \left\| \mathrm{e}^{-\mu\ell(x)}\psi_{\lambda;j} \right\| \ \le\ D_\varepsilon\ . \tag{3.36}$$

In fact, since $\sum_x E_{\{\lambda\}}(x,x) \ = \ \operatorname{tr} E_{\{\lambda\}} \ \le \ C_4$, we may pick $x_0 \in \mathbb{Z}^2$ such that $E_{\{\lambda\}}(x_0,x_0) = \max_x E_{\{\lambda\}}(x,x)$. Let $\psi_{\lambda;0}(x) = E_{\{\lambda\}}(x,x_0)/E_{\{\lambda\}}(x_0,x_0)^{1/2}$.

This normalized eigenfunction satisfies the bounds

$$|\psi_{\lambda;0}(x)| \;\le\; \begin{cases} D_\varepsilon \mathrm{e}^{\varepsilon|x_0|}\mathrm{e}^{-\mu|x-x_0|}/E_{\{\lambda\}}(x_0,x_0)^{1/2}\,, \\ E_{\{\lambda\}}(x_0,x_0)^{1/2}\,. \end{cases}$$

The first one follows from (3.22) for $g(H_B)=E_{\{\lambda\}}$, and the second from

$$\left|E_{\{\lambda\}}(x,x_0)\right| \le E_{\{\lambda\}}(x,x)^{1/2}E_{\{\lambda\}}(x_0,x_0)^{1/2} \le E_{\{\lambda\}}(x_0,x_0)\;.$$

Combining them into a geometric mean yields $|\psi(x)| \le D_\varepsilon^{\frac12}\mathrm{e}^{\frac{\varepsilon}{2}|x_0|}\mathrm{e}^{-\frac{\mu}{2}|x-x_0|}$ and, by the triangle inequality,

$$\left|\psi_{\lambda;0}(x)\overline{\psi}_{\lambda;0}(x')\right| \;\le\; D_\varepsilon \mathrm{e}^{\varepsilon|x_0|}\mathrm{e}^{-\frac{\mu}{2}\left(|x-x_0|+|x'-x_0|\right)} \;\le\; D_\varepsilon \mathrm{e}^{\varepsilon|x|}\mathrm{e}^{-\left(\frac{\mu}{2}-\varepsilon\right)|x-x'|}\;.$$

For small ε the bound (3.22) is reproduced for $\psi_{\lambda;0}\,(\psi_{\lambda;0}\;,\;\cdot\;)$ in place of $E_{\{\lambda\}}$, with a smaller value of μ. Since the rank of $E_{\{\lambda\}}-\psi_{\lambda;0}\,(\psi_{\lambda;0}\;,\;\cdot\;)$ is one less than the rank of $E_{\{\lambda\}}$, the task is completed by induction.

After these preliminaries, we turn to the proof of Lemma 4 proper. We denote by $\widetilde{\mathcal{E}}_\Delta$ the eigenvalues in $\mathcal{E}_\Delta$ listed according to multiplicity. More precisely, we let $\widetilde{\mathcal{E}}_\Delta$ be the set of pairs $\zeta=(\lambda;n)$ with $\lambda\in\mathcal{E}_\Delta$ and n a non-negative integer less than the multiplicity of λ. The eigenvectors $\{\psi_\zeta,\,\zeta\in\widetilde{\mathcal{E}}_\Delta\}$ constructed above are an ortho-normal basis for $\mathrm{ran}E_\Delta$.

Let, for $\zeta\in\widetilde{\mathcal{E}}_\Delta$,

$$M_\zeta \;=\; \min\left(\|\Lambda_1\psi_\zeta\|\,,\|(1-\Lambda_1)\psi_\zeta\|\,,\|\Lambda_2\psi_\zeta\|\,,\|(1-\Lambda_2)\psi_\zeta\|\right)\;.$$

We claim that

$$\sum_{\zeta\in\widetilde{\mathcal{E}}_\Delta} M_\zeta \;<\; \infty\;. \tag{3.37}$$

This states that almost all eigenfunctions are localized in at least one among the left, right, upper, and lower half planes, and hence in at most two (intersecting) ones. In particular almost no eigenfunction encircles the origin, which makes them insensitive to a flux tube applied there —a fact used in some explanations [17,28] of the QHE.

We apply (3.36) to $\psi_\zeta\,(\psi_\zeta,\cdot)$ and use that for rank one operators $\|\phi\,(\psi,\cdot)\|=\|\phi\|\,\|\psi\|$ to obtain $\|\mathrm{e}^{\mu\ell(x)}\mathrm{e}^{-\varepsilon|x|}\psi_\zeta\|\|\mathrm{e}^{-\mu\ell(x)}\psi_\zeta\|\le D_\varepsilon$. For $\ell(x)=x_1$ we have $\Lambda_1(x)\le\mathrm{e}^{-\mu\ell(x)}$, implying

$$C_2\,\|\Lambda_1\psi_\zeta\|^{-1} \;\ge\; \left\|\mathrm{e}^{\mu x_1}\mathrm{e}^{-\varepsilon|x|}\psi_\zeta\right\|\;,$$

similar estimates for $1-\Lambda_1$, Λ_2, and $1-\Lambda_2$ have x_1 on the r.h.s. replaced by $-x_1$, x_2, and $-x_2$ respectively. Therefore,

$$\begin{aligned} M_\zeta^{-2} \;&=\; \max\left(\|\Lambda_1\psi_\zeta\|^{-2},\|(1-\Lambda_1)\psi_\zeta\|^{-2},\|\Lambda_2\psi_\zeta\|^{-2},\|(1-\Lambda_2)\psi_\zeta\|^{-2}\right) \\ &\ge\; \frac14\left(\|\Lambda_1\psi_\zeta\|^{-2}+\|(1-\Lambda_1)\psi_\zeta\|^{-2}+\|\Lambda_2\psi_\zeta\|^{-2}+\|(1-\Lambda_2)\psi_\zeta\|^{-2}\right) \\ &\ge\; \frac{1}{4C_2^2}\left(\psi_\zeta,\mathrm{e}^{-2\varepsilon|x|}\left(\mathrm{e}^{2\mu x_1}+\mathrm{e}^{-2\mu x_1}+\mathrm{e}^{2\mu x_2}+\mathrm{e}^{-2\mu x_2}\right)\psi_\zeta\right) \\ &\ge\; \frac{1}{4C_2^2}\left(\psi_\zeta,\mathrm{e}^{(\mu-2\varepsilon)|x|}\psi_\zeta\right), \end{aligned}$$

where we use $\mathrm{e}^{2\mu|x_1|} + \mathrm{e}^{2\mu|x_2|} \geq \mathrm{e}^{\mu(|x_1|+|x_2|)}$. Now let $\varepsilon > 0$ be small enough that $\delta := \mu - 2\varepsilon > 0$. Then

$$M_\zeta \;\leq\; 2D_\varepsilon \left[\left(\psi_\zeta, \mathrm{e}^{\delta|x|}\psi_\zeta\right)\right]^{-\frac{1}{2}} \;\leq\; 2D_\varepsilon \left(\psi_\zeta, \mathrm{e}^{-\frac{1}{2}\delta|x|}\psi_\zeta\right),$$

where in the last step we have applied Jensen's inequality with the convex function $t \mapsto t^{-\frac{1}{2}}$. As $\{\psi_\zeta : \zeta \in \widetilde{\mathcal{E}}_\Delta\}$ are ortho-normal, we conclude that

$$\sum_{\zeta \in \widetilde{\mathcal{E}}_\Delta} M_\zeta \;\leq\; 2D_\varepsilon \operatorname{tr} \mathrm{e}^{-\frac{1}{2}\delta|x|} \;<\; \infty,$$

proving (3.37).

We can now estimate the traces in (2.26):

$$\left|\operatorname{tr} E_{\{\lambda\}} \mathrm{e}^{-\mathrm{i}H_B t} \left[H_B, \Lambda_1\right] \mathrm{e}^{\mathrm{i}H_B t} \Lambda_2 E_{\{\lambda\}}\right| \;\leq\; \sum_{\zeta=(\lambda;\cdot)} \left|\left(\psi_\zeta, \left[H_B, \Lambda_1\right] \mathrm{e}^{\mathrm{i}H_B t} \Lambda_2 \psi_\zeta\right)\right| \;. \tag{3.38}$$

By inserting $\Lambda_2 = 1 - (1 - \Lambda_2)$, the terms on the right hand side may also be expressed as

$$\left|\left(\psi_\zeta, \left[H_B, \Lambda_1\right] \mathrm{e}^{\mathrm{i}H_B t}(1 - \Lambda_2)\psi_\zeta\right)\right| \;.$$

Using

$$\left(\psi_\zeta, \left[H_B, \Lambda_1\right] \phi\right) \;=\; \left(\Lambda_1\psi_\zeta, (\lambda - H_B)\phi\right) \;=\; -\left((1 - \Lambda_1)\psi_\zeta, (\lambda - H_B)\phi\right) \;,$$

one sees that (3.38) is bounded by a constant times $\sum_{\zeta=(\lambda;\cdot)} M_\zeta$, so the right hand side of (2.26) is bounded by $\sum_\zeta M_\zeta$. $\square$

4. Analysis of the Harper Hamiltonian

In this section we prove Theorem 3 which shows that the contribution from bulk states in (1.12) can be non-zero. We begin with the following proposition:

Proposition 1. *Let $f(\{V_x\}_{x\in\mathbb{Z}^d})$ be a function which is bounded and continuous in the product topology on $\left\{\{V_x\}_{x\in\mathbb{Z}^d} \,|\, \operatorname{Im} V_x \leq 0\right\} \;=\; \overline{\mathbb{C}}_-^{\;\mathbb{Z}^d}$. If f is separately analytic in each V_x, then*

$$\mathbb{E}\,(f) \;=\; f(\{-\mathrm{i}\}_{x\in\mathbb{Z}^d}) \;, \tag{4.1}$$

where $\mathbb{E}\,(\cdot)$ represents the average with respect to the product measure

$$\mathrm{d}\mathbb{P}(\{V_x\}_{x\in\mathbb{Z}^d}) \;:=\; \prod_{x\in\mathbb{Z}^d} \frac{\mathrm{d}V_x}{\pi(1 + V_x^2)} \;,$$

supported on $\left\{\{V_x\}_{x\in\mathbb{Z}^d} \,|V_x \in \mathbb{R}\right\} = \mathbb{R}^{\mathbb{Z}^d}$. The same statement holds for $\mathbb{C}_+$, $+\mathrm{i}$ in place of $\mathbb{C}_-$, $-\mathrm{i}$.

Proof. Let S_j be an increasing sequence of finite sets with $\lim_j S_j = \cup_j S_j = \mathbb{Z}^d$, and let $\mathcal{F}_j^c$ denote the σ-algebra generated by $\{V_x\}_{x\in S_j^c}$. So conditional expectation with respect to $\mathcal{F}_j^c$ is given by "averaging out" the variables $\{V_x\}_{x\in S_j}$. Thus

$$f_j(\{V_x\}_{x\in S_j^c}) \;:=\; \mathbb{E}\left(f|\mathcal{F}_j^c\right) \;=\; \int \prod_{x\in S_j} \frac{\mathrm{d}V_x}{\pi(1+V_x^2)} f(\{V_x\}_{x\in S_j} \times \{V_x\}_{x\in S_j^c}) \,.$$

Because f is bounded and separately analytic in each V_x, we may evaluate the integrals on the right hand side by residues to obtain

$$f_j(\{V_x\}_{x\in S_j^c}) \;=\; f(\{-\mathrm{i}\}_{x\in S_j} \times \{V_x\}_{x\in S_j^c}) \,.$$

Because f is continuous and $\lim_{j\to\infty} \{-\mathrm{i}\}_{x\in S_j} \times \{V_x\}_{x\in S_j^c} = \{-\mathrm{i}\}_{x\in\mathbb{Z}^d}$ in the product topology on $\overline{\mathbb{C}_-}^{\mathbb{Z}^d}$, we have

$$\lim_{j\to\infty} f_j(\{V_x\}_{x\in S_j^c}) \;=\; f(\{-\mathrm{i}\}_{x\in\mathbb{Z}^d})$$

for any $\{V_x\}_{x\in\mathbb{Z}^d} \in \mathbb{R}^{\mathbb{Z}^d}$. Since f_j are uniformly bounded and $\mathbb{E}(f_j) = \mathbb{E}(f)$ for every j, we conclude by dominated convergence that (4.1) holds. $\square$

Turning now to the proof of Theorem 3, we first recall that, by Lemma 1,

$$-\frac{\mathrm{i}}{2} \lim_{a\to\infty} \mathrm{tr}\, \rho'(H_a) \{[H_a, \Lambda_1]\,,\, \Lambda_2\} \;=\; \mathrm{Re}\,\mathrm{tr}\, \Sigma_B''(0)\,, \tag{4.2}$$

where

$$\mathrm{i}\Sigma_B''(0) \;=\; -\frac{1}{2\pi} \int \mathrm{d}m(z) \partial_{\bar z}\rho(z)\, \mathrm{tr}\, \underbrace{R_B(z)\,[H_\phi, \Lambda_1]\, R_B(z)\,[H_\phi, \Lambda_2]\, R_B(z)}_{T_B(z)} \,.$$

In going from (2.10) to the above expression for $\Sigma_B''(0)$ we have replaced H_B by H_ϕ in the commutators $[H_B, \Lambda_i]$ since the random potential commutes with each switch function Λ_i.

By Lemma 1, we have $\sup_a |\mathrm{tr}\,\rho'(H_a)\{\,[H_a, \Lambda_1]\,, \Lambda_2\}| \;\le\; C \;<\; \infty$, with a constant C that depends on ρ and on the bounds C_1, C_3 in (1.1, 1.6), but <u>not</u> on the random constant C_2 in (1.2). Since the constants C_1, C_3 are non-random in our setup, the expectation in (1.22) is well defined, and furthermore can be exchanged with the limit.

We claim that for $\mathrm{Im}\, z \neq 0$

$$\mathbb{E}\,(\mathrm{tr}\, T_B(z)) \;=\; \mathrm{tr}\, T_\phi(z + \mathrm{i}\alpha\sigma(z))\,, \tag{4.3}$$

where $T_\phi(z) = R_\phi(z)\,[H_\phi, \Lambda_1]\, R_\phi(z)\,[H_\phi, \Lambda_2]\, R_\phi(z)$, with $R_\phi(z) \;=\; (H_\phi - z)^{-1}$, and $\sigma(z) = \mathrm{Im}\, z/|\,\mathrm{Im}\, z|$ denotes the sign of the imaginary part of z. Indeed, for $\mathrm{Im}\, z > 0$, it suffices to verify that $f_z(\{V_x\}) \;=\; \mathrm{tr}\, T_B(z)$ obeys the hypotheses of Proposition 1. For that purpose, it is useful to note that

$$G_z(\{V_x\}_{x\in\mathbb{Z}^d}) \;:=\; (H_\phi + \alpha V - z)^{-1}$$

is a continuous map from $\big\{\{V_x\}_{x\in\mathbb{Z}^d}\,|\,\mathrm{Im}\,V_x\le 0\big\}$ to the bounded operators on $\ell^2(\mathbb{Z}^2)$ endowed with the strong operator topology. Indeed, z is in the resolvent set of $H_\phi+\alpha V$ since the numerical range of this operator is contained in the closed lower half plane. Thus G_z is well defined, SOT-continuous (since $\{V_x\}_x\mapsto H_\phi+\alpha V$ and $A\mapsto A^{-1}$ are SOT-continuous), and

$$\left\|G_z(\{V_x\}_{x\in\mathbb{Z}^d})\right\| \;\le\; \frac{1}{\mathrm{dist}(z,\mathrm{num.\ range}(H_\phi+\alpha V))} \;\le\; \frac{1}{|\,\mathrm{Im}\,z|}\,. \tag{4.4}$$

Furthermore, the Combes-Thomas bound (3.2) extends to G_z, i.e.,

$$\left\|\mathrm{e}^{\delta\ell(x)}G_z(\{V_x\}_{x\in\mathbb{Z}^d})\mathrm{e}^{-\delta\ell(x)}\right\| \;\le\; \frac{C}{|\,\mathrm{Im}\,z|}\,,\quad \delta^{-1}=C\left(1+|\,\mathrm{Im}\,z|^{-1}\right)\,, \tag{4.5}$$

with $\ell(x)$ as in (3.4). The resolvent of $\mathrm{e}^{\pm\delta\ell(x)}(H_\phi+\alpha V)\mathrm{e}^{\mp\delta\ell(x)}$, considered as a perturbation of $H_\phi+\alpha V$, is in fact as stable as in (3.2) where H_ϕ was self-adjoint, since the same bound (4.4) still holds for $\mathrm{Im}\,z>0$. Furthermore, we see in this way that

$$\{V_x\}_{x\in\mathbb{Z}^d}\mapsto \mathrm{e}^{\delta\ell(x)}G_z(\{V_x\}_{x\in\mathbb{Z}^d})\mathrm{e}^{-\delta\ell(x)}$$

is SOT-continuous.

Thus, for $\mathrm{Im}\,z>0$,

$$\begin{aligned}\mathrm{tr}\,T_B(z) \;=\; &\mathrm{tr}\,G_z(\{V_x\}_{x\in\mathbb{Z}^d})\,[H_\phi,\Lambda_1]\,\mathrm{e}^{\delta|x_1|}\;\cdot\;\mathrm{e}^{-\delta|x_1|}G_z(\{V_x\}_{x\in\mathbb{Z}^d})\mathrm{e}^{\delta|x_1|}\;\cdot\\ &\cdot\;\mathrm{e}^{-\delta(|x_1|+|x_2|)}\;\cdot\mathrm{e}^{\delta|x_2|}\,[H_\phi,\Lambda_2]\,G_z(\{V_x\}_{x\in\mathbb{Z}^d})\,,\end{aligned}$$

is a continuous function, which is bounded by

$$\begin{aligned}|\mathrm{tr}\,T_B(z)| \;\le\;& \frac{C}{\delta^2}\;\left\|G_z(\{V_x\}_{x\in\mathbb{Z}^d})\right\|^2\,\left\|[H_\phi,\Lambda_1]\,\mathrm{e}^{\delta|x_1|}\right\|\cdot\\ &\cdot\left\|\mathrm{e}^{-\delta|x_1|}G_z(\{V_x\}_{x\in\mathbb{Z}^d})\mathrm{e}^{\delta|x_1|}\right\|\,\left\|\mathrm{e}^{\delta|x_2|}\,[H_\phi,\Lambda_2]\right\|\\ \le\;& C\;\frac{(1+|\,\mathrm{Im}\,z|^{-1})^2}{|\,\mathrm{Im}\,z|^3}\,,\end{aligned} \tag{4.6}$$

with the factor of $1/\delta^2$ coming from the estimate (3.9) on the trace of $\mathrm{e}^{-\delta|x|}$. A similar argument is used for $\mathrm{Im}\,z<0$. Since the separate analyticity of $f_z(\cdot)=\mathrm{tr}\,T_B(z)$ is clear, Proposition 1 applies.

We see that

$$\mathbb{E}\left(\mathrm{Re}\,\mathrm{tr}\,\Sigma_B''(0)\right) \;=\; -\frac{1}{2\pi}\,\mathrm{Im}\int\mathrm{d}m(z)\partial_{\bar z}\rho(z)\,\mathrm{tr}\,T_\phi(z+\mathrm{i}\alpha\sigma(z))\,, \tag{4.7}$$

where the interchange of $\int\mathrm{d}m(z)$ and $\mathbb{E}$ is justified by Fubini's theorem and (4.6) since we may arrange for $\partial_{\bar z}\rho(z)$ to vanish faster than $|\,\mathrm{Im}\,z|^5$ as z approaches the real axis. We note that

$$|\mathrm{tr}\,T_\phi(z+\mathrm{i}\alpha\sigma(z))| \;\le\; \frac{C_\alpha}{[x^2+(|y|+\alpha)^2]^{3/2}}\,. \tag{4.8}$$

In fact, now that $V = 0$, $|\operatorname{Im} z|^{-1}$ in (4.4) may be replaced by $\operatorname{dist}(z, \sigma(H_\phi))^{-1} \leq \operatorname{dist}(z, [-2, 2])^{-1}$ and the same replacement carries over to the denominator in the estimate (4.6) for $\operatorname{tr} T_\phi(z)$.

The only singularities in the integrand on the right hand side of (4.7) are jump discontinuities at $\operatorname{Im} z = 0$. Integrating by parts, on the upper and lower half planes separately, we find

$$\mathbb{E}\left(\operatorname{Re}\operatorname{tr} \Sigma_B''(0)\right) \;=\; \frac{1}{2\pi}\operatorname{Re}\int_{-\infty}^{\infty} \mathrm{d}x\rho(x)\operatorname{tr}\left(T_\phi(x+\alpha\mathrm{i}) - T_\phi(x-\alpha\mathrm{i})\right)\;, \qquad (4.9)$$

since by (4.8) there are no contributions from the boundary at infinity. Upon writing $\rho(x) = -\int_x^\infty \rho'(\lambda)\mathrm{d}\lambda$, and interchanging λ and x integration we obtain

$$\mathbb{E}\left(\operatorname{Re}\operatorname{tr} \Sigma_B''(0)\right) \;=\; -\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\lambda\rho'(\lambda)\int_{-\infty}^{\lambda} \operatorname{Re}\operatorname{tr}\left(T_\phi(x+\alpha\mathrm{i}) - T_\phi(x-\alpha\mathrm{i})\right)\mathrm{d}x\;. \qquad (4.10)$$

This proves (1.22) with

$$j_B(\lambda) \;=\; \frac{1}{2\pi}\operatorname{Re}\int_{-\infty}^{\lambda} \operatorname{tr}\left(T_\phi(x+\alpha\mathrm{i}) - T_\phi(x-\alpha\mathrm{i})\right)\mathrm{d}x\;. \qquad (4.11)$$

To obtain the asymptotic expression (1.23), note that for $|\lambda| > 2$

$$j_B(\lambda) \;=\; \frac{1}{2\pi}\operatorname{Re}\int_{-\alpha}^{\alpha} \mathrm{i}\mathrm{d}\eta\operatorname{tr} T_\phi(\lambda+\mathrm{i}\eta)\;, \qquad (4.12)$$

because the difference of the right hand sides of (4.11, 4.12) is the real part of an integral around a closed contour, which may be deformed to infinity, of the analytic function $\operatorname{tr} T_\phi(z)$, which vanishes like $1/|z|^2$ as $z \to \infty$. (It is of interest to note that for λ in an internal gap of the spectrum of H_ϕ, the corresponding contour integral gives the Bulk conductance $\sigma_B^{(\phi)}(\lambda)$ for the Hamiltonian H_ϕ at Fermi energy λ, so $j_B(\lambda) = \sigma_B^{(\phi)}(\lambda) + \frac{1}{2\pi}\operatorname{Re}\mathrm{i}\int_{-\alpha}^{\alpha}\mathrm{d}\eta\operatorname{tr} T_\phi(\lambda+\mathrm{i}\eta)$.)

It is useful to rewrite (4.12) as

$$j_B(\lambda) \;=\; \frac{1}{2\pi}\operatorname{Re}\int_0^{\alpha} \mathrm{i}\mathrm{d}\eta\left(\operatorname{tr} T_\phi(\lambda+\mathrm{i}\eta) - \overline{\operatorname{tr} T_\phi(\lambda-\mathrm{i}\eta)}\right)\;, \qquad (4.13)$$

which follows by considering the contributions from $\eta < 0$ and $\eta > 0$ separately, and using $\operatorname{Re}\mathrm{i}\,w = -\operatorname{Re}\mathrm{i}\,\overline{w}$.

We obtain (1.23) from the series for $T_B(\lambda+\mathrm{i}\eta) - \overline{\operatorname{tr} T_B(\lambda-\mathrm{i}\eta)}$ produced by expanding each resolvent in a Neumann series. For sufficiently large $|\lambda|$,

$$R_\phi(\lambda+\mathrm{i}\eta) \;=\; -\frac{1}{\lambda}\sum_{n=0}^{\infty}\left[\frac{H_\phi - \mathrm{i}\eta}{\lambda}\right]^n \qquad (4.14)$$

is absolutely convergent, and

$$\begin{aligned}\operatorname{tr} T_\phi(\lambda+\mathrm{i}\eta) \;=\; -\frac{1}{\lambda^3}\sum_{N=0}^{\infty}\frac{1}{\lambda^N}\sum_{n_1+n_2+n_3=N} &\operatorname{tr}\left(H_\phi - \mathrm{i}\eta\right)^{n_1}\left[H_\phi, \Lambda_1\right]\cdot \\ &\cdot\left(H_\phi - \mathrm{i}\eta\right)^{n_2}\left[H_\phi, \Lambda_2\right]\left(H_\phi - \mathrm{i}\eta\right)^{n_3}\;,\end{aligned}$$

To prove convergence here, it is useful to note that in addition to (4.14), the series

$$\mathrm{e}^{\delta|x|}R_\phi(\lambda+\mathrm{i}\eta)\mathrm{e}^{-\delta|x|} \;=\; -\frac{1}{\lambda}\sum_{n=0}^{\infty}\left[\frac{\mathrm{e}^{\delta|x|}H_\phi\mathrm{e}^{-\delta|x|}-\mathrm{i}\eta}{\lambda}\right]^n$$

is also absolutely convergent, in light of (1.1).

By cyclicity of the trace

$$\begin{aligned}\operatorname{tr}T_\phi(\lambda+\mathrm{i}\eta) \;=\; -\sum_{N=0}^{\infty}\frac{1}{\lambda^{N+3}}\sum_{n=0}^{N}(n+1)\operatorname{tr}(H_\phi-\mathrm{i}\eta)^n\,[H_\phi,\Lambda_1]\cdot\\ \cdot(H_\phi-\mathrm{i}\eta)^{N-n}\,[H_\phi,\Lambda_2]\;,\end{aligned}$$

and, making use of the identity $\overline{\operatorname{tr}T}=\operatorname{tr}T^*$,

$$\begin{aligned}\overline{\operatorname{tr}T_\phi(\lambda-\mathrm{i}\eta)} \;=\; -\sum_{N=0}^{\infty}\frac{1}{\lambda^{N+3}}\sum_{n=0}^{N}(N-n+1)\operatorname{tr}(H_\phi-\mathrm{i}\eta)^n\,[H_\phi,\Lambda_1]\cdot\\ \cdot(H_\phi-\mathrm{i}\eta)^{N-n}\,[H_\phi,\Lambda_2]\;.\end{aligned}$$

Thus

$$\begin{aligned}&\operatorname{tr}T_\phi(\lambda+\mathrm{i}\eta)-\overline{\operatorname{tr}T_\phi(\lambda-\mathrm{i}\eta)}\\ &=\;-\sum_{N=0}^{\infty}\frac{1}{\lambda^{N+3}}\sum_{n=0}^{N}(2n-N)\operatorname{tr}(H_\phi-\mathrm{i}\eta)^n\,[H_\phi,\Lambda_1]\cdot(H_\phi-\mathrm{i}\eta)^{N-n}\,[H_\phi,\Lambda_2]\;,\end{aligned}$$

which is the desired expansion.

The first term ($N=0$) of this series vanishes trivially. The second ($N=1$) also vanishes, because

$$\begin{aligned}\operatorname{tr}[H_\phi,\Lambda_1]\,(H_\phi-\mathrm{i}\eta)\,[H_\phi,\Lambda_2]-\operatorname{tr}(H_\phi-\mathrm{i}\eta)\,[H_\phi,\Lambda_1]\,[H_\phi,\Lambda_2]&\\ =\;-\operatorname{tr}[H_\phi,[H_\phi,\Lambda_1]]\,[H_\phi,\Lambda_2]&\\ =\;-\sum_x\sum_y[H_\phi,[H_\phi,\Lambda_1]]\,(x,y)\,[H_\phi,\Lambda_2]\,(y,x)\;=\;0\;,&\end{aligned}\tag{4.15}$$

since $[H_\phi,\Lambda_2]\,(y,x)\neq0$ only for $|x-y|=1$ and $[H_\phi,[H_\phi,\Lambda_1]]\,(x,y)\neq0$ only for $|x-y|=0,2$ as only nearest neighbor hopping terms are present in H_ϕ. However the coefficient of λ^{-5} ($N=2$) is non-zero, and given by

$$\begin{aligned}2\operatorname{tr}[H_\phi,\Lambda_1]\,(H_\phi-\mathrm{i}\eta)^2\,[H_\phi,\Lambda_2]-2\operatorname{tr}(H_\phi-\mathrm{i}\eta)^2\,[H_\phi,\Lambda_1]\,[H_\phi,\Lambda_2]&\\ =\;2\operatorname{tr}[H_\phi,\Lambda_1]\,H_\phi^2\,[H_\phi,\Lambda_2]-2\operatorname{tr}H_\phi^2\,[H_\phi,\Lambda_1]\,[H_\phi,\Lambda_2]&\\ =\;-2\operatorname{tr}H_\phi^2\,[[H_\phi,\Lambda_1]\,,[H_\phi,\Lambda_2]]\;,&\end{aligned}$$

since the term proportional to η vanishes by (4.15) and the term proportional to η^2 is the trace of a commutator, $\operatorname{tr}[[H_\phi,\Lambda_1]\,,[H_\phi,\Lambda_2]]=0$.

To calculate this term explicitly, recall that $\Lambda_i = I[x_i < 0]$ so, by (1.21),

$$\begin{aligned}
[H_\phi, \Lambda_1]\,(x,x') &= (\Lambda_1(x') - \Lambda_1(x))H_\phi(x,x') \\
&= \begin{cases} 1\,, & x = (0,x_2)\,,\ x' = (-1,x_2)\,, \\ -1\,, & x = (-1,x_2)\,,\ x' = (0,x_2)\,, \\ 0\,, & \text{all other } x, x', \end{cases}
\end{aligned}$$

which is more succinctly expressed in Dirac notation:

$$[H_\phi, \Lambda_1] \;=\; \sum_{a\in\mathbb{Z}} |0,a\rangle\,\langle -1,a| - |-1,a\rangle\,\langle 0,a|\ .$$

Similarly,

$$[H_\phi, \Lambda_2] \;=\; \sum_{a\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}\phi a}\,|a,0\rangle\,\langle a,-1| - \mathrm{e}^{-\mathrm{i}\phi a}\,|a,-1\rangle\,\langle a,0|\ .$$

Thus

$$\begin{aligned}
[[H_\phi, \Lambda_1]\,, [H_\phi, \Lambda_2]] &= (\mathrm{e}^{-\mathrm{i}\phi} - 1)\Big(\,|0,0\rangle\,\langle -1,-1| + |-1,0\rangle\,\langle 0,-1|\,\Big) \\
&\quad - (\mathrm{e}^{\mathrm{i}\phi} - 1)\Big(\,|0,-1\rangle\,\langle -1,0| + |-1,-1\rangle\,\langle 0,0|\,\Big)\,,
\end{aligned}$$

and

$$\begin{aligned}
&\operatorname{tr} H_\phi^2\,[[H_\phi, \Lambda_1]\,, [H_\phi, \Lambda_2]] \\
&\qquad = (\mathrm{e}^{-\mathrm{i}\phi} - 1)\Big(\langle -1,-1|\, H_\phi^2\,|0,0\rangle + \langle 0,-1|\,H_\phi^2\,|-1,0\rangle\Big) \;-\; c.c.\,.
\end{aligned}$$

Finally, since

$$\langle -1,-1|\,H_\phi^2\,|0,0\rangle \;=\; 1 + \mathrm{e}^{-\mathrm{i}\phi}\,, \quad \langle 0,-1|\,H_\phi^2\,|-1,0\rangle \;=\; 1 + \mathrm{e}^{\mathrm{i}\phi}\,,$$

we have

$$\begin{aligned}
2\operatorname{tr} H_\phi^2\,[[H_\phi, \Lambda_1]\,, [H_\phi, \Lambda_2]] &= 4(\mathrm{e}^{-\mathrm{i}\phi} - 1)\,(\cos(\phi) + 1) \;-\; c.c. \\
&= -8\mathrm{i}\sin(\phi)(\cos(\phi) + 1)\,.
\end{aligned}$$

Therefore

$$\operatorname{tr} T_\phi(\lambda + \mathrm{i}\eta) - \overline{\operatorname{tr} T_\phi(\lambda - \mathrm{i}\eta)} \;=\; 8\mathrm{i}\sin(\phi)(\cos(\phi)+1)\lambda^{-5} \;+\; \mathcal{O}(\lambda^{-6})\,,$$

and

$$j_B(\lambda) \;=\; -\frac{4\alpha}{\pi}\sin(\phi)(\cos(\phi)+1)\lambda^{-5} \;+\; \mathcal{O}(\lambda^{-6})\,,$$

which gives (1.23). This completes the proof of Theorem 3. $\square$

Appendix: conductance plateaus

Localization is an essential prerequisite for the QHE. Some localization condition, valid at energies in an interval Δ, is proven and used in [6,2]. It ensures that $\sigma_B(\lambda)$ is

1. well defined as given by (1.4),
2. constant in $\lambda \in \Delta$, and
3. $2\pi\sigma_B(\lambda) \in \mathbb{Z}$.

These results also rest on a homogeneity assumption for the Hamiltonian H_B, or on its Fermi projections P_λ, namely that they be invariant or ergodic under magnetic translations. The purpose of the Appendix is to establish (1.-3.) under assumptions (1.1-1.3), which do not entail translation invariance.

Proposition 2. *Assume (1.1) and (1.2). Then $\sigma_B(\lambda)$ is well-defined. If in addition (1.3) holds, then $\sigma_B(\lambda)$ is constant in $\lambda \in \Delta$.*

Proposition 3. *Assume (1.1) and (1.2). Then $2\pi\sigma_B(\lambda) \in \mathbb{Z}$ for $\lambda \in \Delta$.*

We remark that here constancy is proven without combining integrality and continuity.

A.1. Proof of Prop. 2. We consider Borel sets $S \subset \mathbb{R}$ that either contain or are disjoint from $\{\lambda | \lambda < \Delta\}$ and similarly for $\{\lambda | \lambda > \Delta\}$. The class of such sets S is closed under unions and complements. We associate a bulk Hall conductance to S by setting

$$\begin{aligned} \sigma_B(S) &= -\mathrm{i}\,\mathrm{tr}\, E_S\left[[E_S, \Lambda_1], [E_S, \Lambda_2]\right] \\ &= \mathrm{i}\,\mathrm{tr}\left(E_S \Lambda_1 E_S^\perp \Lambda_2 E_S - E_S \Lambda_2 E_S^\perp \Lambda_1 E_S\right) , \end{aligned} \tag{A.1}$$

where $E_S^\perp = 1 - E_S$ and the second line follows from

$$E_S\left[E_S, \Lambda_1\right] = E_S\left[E_S, \Lambda_1\right] E_S^\perp = E_S \Lambda_1 E_S^\perp \; .$$

Note that $\sigma_B(\lambda_0) = \sigma_B((-\infty, \lambda_0))$. We claim that, if $S_1 \cap S_2 = \emptyset$, then

$$E_{S_1} \Lambda_1 E_{S_2} \Lambda_2 E_{S_1} \in \mathfrak{I}_1 \; , \tag{A.2}$$

$$\sigma_B(S_1 \cup S_2) = \sigma_B(S_1) + \sigma_B(S_2) \; , \tag{A.3}$$

and moreover

$$\lim_{n\to\infty} \sigma_B(S_n) = 0 \text{ if } S_n \downarrow \emptyset \; . \tag{A.4}$$

In particular, (A.2) and its adjoint for $S_1 = S$, $S_2 = \mathbb{R} \setminus S$ imply that the two terms in the final expression of (A.1) are separately trace class.

(A.2): In the factorization

$$E_{S_1} \Lambda_1 E_{S_2} \Lambda_2 E_{S_1} = E_{S_1} \Lambda_1 E_{S_2} \mathrm{e}^{3\delta|x_1|} \mathrm{e}^{-\delta|x|} \cdot \mathrm{e}^{-\delta|x|} \cdot \mathrm{e}^{-\delta|x|} \mathrm{e}^{3\delta|x_2|} E_{S_2} \Lambda_2 E_{S_1} \; , \tag{A.5}$$

the middle $\mathrm{e}^{-\delta|x|} = \mathrm{e}^{-\delta|x_1|}\mathrm{e}^{-\delta|x_2|}$ is trace class by (3.9), so that we need to show

$$\left\| E_{S_1} \Lambda_i E_{S_2} \mathrm{e}^{3\delta|x_i|} \mathrm{e}^{-\delta|x|} \right\| \;<\; \infty\,, \quad (i=1,2)\,. \tag{A.6}$$

This follows from (3.25, 3.27) and part (iii) of Lemma 6, with a bound which is uniform in S_1, S_2.

(A.4): By (A.1, A.5) and (2.21) it suffices to show

$$E_{S_n} \Lambda_i E_{S_n}^{\perp} \mathrm{e}^{3\delta|x_i|} \mathrm{e}^{-\delta|x|} \;\xrightarrow[n\to\infty]{s}\; 0\,.$$

Since the l.h.s. is uniformly bounded in norm by the remark just made, we may drop the exponentials as explained in connection with (3.16, 3.17). Then the claim becomes obvious.

(A.3): From $E_{S_1\cup S_2} = E_{S_1} + E_{S_2}$ and (2.2) we have

$$\sigma_B(S_1\cup S_2) \;=\; \sum_{i=1}^{2} \left(\operatorname{tr} E_{S_i} \Lambda_i E_{S_1\cup S_2}^{\perp} \Lambda_2 E_{S_i} - \operatorname{tr} E_{S_1\cup S_2}^{\perp} \Lambda_1 E_{S_i} \Lambda_2 E_{S_1\cup S_2}^{\perp} \right)\,.$$

We use $E_{S_1\cup S_2}^{\perp} = E_{S_i}^{\perp} - E_{S_{i+1}}$ (with $i+1$ defined mod 2) and obtain

$$\begin{aligned} \sigma_B(S_1\cup S_2) \;&=\; \sum_{i=1}^{2} \sigma_B(S_i) - \sum_{i=1}^{2} \operatorname{tr} E_{S_i} \Lambda_1 E_{S_{i+1}} \Lambda_2 E_{S_i} \\ &\quad + \sum_{i=1}^{2} \operatorname{tr} E_{S_{i+1}} \Lambda_1 E_{S_i} \Lambda_2 E_{S_{i+1}} \\ &=\; \sigma_B(S_1) + \sigma_B(S_2)\,. \end{aligned}$$

We finally prove constancy by showing that $\sigma_B([a,b]) = 0$ for any $[a,b] \subset \Delta$. Since $\sigma(H_B)$ is pure point in Δ we have

$$E_n \;:=\; \sum_{i=1}^{n} E_{\{\lambda_i\}} \;\xrightarrow[n\to\infty]{s}\; E_{\mathcal{E}_{[a,b]}} \;=\; E_{[a,b]}\,,$$

where λ_i is any labeling of the eigenvalues $\lambda \in \mathcal{E}_{[a,b]}$. Now E_n is a finite dimensional projection by (1.3), whence the two terms in

$$\sigma_B\left(\cup_{i=1}^{n}\{\lambda_i\}\right) \;=\; -\mathrm{i}\operatorname{tr}\left(E_n\Lambda_1E_n\Lambda_2E_n - E_n\Lambda_2E_n\Lambda_1E_n\right) \;=\; 0$$

are separately trace class. They cancel by (2.2). We conclude by (A.3, A.4) that

$$\sigma_B([a,b]) \;=\; \sigma_B\left(\cup_{i=1}^{n}\{\lambda_i\}\right) \;+\; \sigma_B\left(\mathcal{E}_{[a,b]} \setminus \cup_{i=1}^{n}\{\lambda_i\}\right) \;\xrightarrow[n\to\infty]{}\; 0\,. \quad \square$$

A.2. Proof of Prop. 3. As in [5] we are going to establish that $2\pi\sigma_B(\lambda)$ is an integer by relating it to the index of a pair of projections.

We first allow the functions Λ_i in (1.4) to switch values at points other than the origin. Let $p = (p_1, p_2) \in \mathbb{Z}^{2*} = \mathbb{Z}^2 + (\frac{1}{2}, \frac{1}{2})$ be the center of a plaquette and set

$$\begin{aligned} \sigma_p &= -\mathrm{i}\,\mathrm{tr}\, P_\lambda \left[[P_\lambda, \Lambda_{1,p}], [P_\lambda, \Lambda_{2,p}]\right] \\ &= \mathrm{i}\,\mathrm{tr}\left([P_\lambda, \Lambda_{1,p}]\, P_\lambda^\perp\, [P_\lambda, \Lambda_{2,p}] - [P_\lambda, \Lambda_{2,p}]\, P_\lambda^\perp\, [P_\lambda, \Lambda_{1,p}]\right) , \end{aligned} \tag{A.7}$$

where $\Lambda_{i,p} = \Lambda(x_i - p_i)$, $(i = 1, 2)$. (Since $\Lambda(n) = \Lambda(n + \frac{1}{2})$ for $n \in \mathbb{Z}$, $\sigma_B(\lambda)$ is just σ_p for $p = -(\frac{1}{2}, \frac{1}{2})$.)

To define the index, let $\theta_p(x) = \arg(x - p)$ be the angle of sight of $x \in \mathbb{Z}^2$ from p, and set $U_p(x) = \mathrm{e}^{\mathrm{i}\theta_p(x)}$. The relevant index is $N_p = \mathrm{Ind}(U_p P_\lambda U_p^*, P_\lambda)$, where $\mathrm{Ind}(P, Q)$ denotes the index of a pair of projections introduced in ref. [5]:

$$\mathrm{Ind}(P, Q) := \dim \mathrm{ran}\, P \cap \ker Q - \dim \mathrm{ran}\, Q \cap \ker P . \tag{A.8}$$

We recall the following basic properties of $\mathrm{Ind}(\cdot, \cdot)$:

1. If $P - Q$ is compact, $\mathrm{Ind}(P, Q)$ is well defined and finite.
2. If $(P - Q)^{2n+1}$ is trace class for some integer $n \geq 0$, then

$$\mathrm{tr}(P - Q)^{2n+1} = \mathrm{Ind}(P, Q) . \tag{A.9}$$

Since N_p is an integer by (A.8), Prop. 3 is a consequence of the identity

$$2\pi\sigma_B(\lambda) = N_p ,$$

to be proved below. Indeed, this is the same strategy employed in refs. [5,2]. The starting point for our proof is the observation that σ_p and N_p are independent of p even without ergodicity for the underlying projection.

Lemma 7. *The index N_p is well defined for any $p \in \mathbb{Z}^{2*}$, and for any $a \in \mathbb{Z}^2$*

i) $N_{p+a} = N_p$,
ii) $\sigma_{p+a} = \sigma_p$.

Proof. Part (i) follows from [5, Prop. 3.8] once we verify that N_p is well defined. For this we follow [2] and show that $(P_\lambda - U_p P_\lambda U_p^*)^3$ is trace class, using

Lemma ([2, Lemma 1]). *For an operator with the matrix elements $T_{x,y}$*

$$\|T\|_3 \equiv (\mathrm{tr}\,|T|^3)^{1/3} \leq \sum_b \left(\sum_x |T_{x+b,x}|^3\right)^{1/3} .$$

In our case, with $T = P_\lambda - U_p P_\lambda U_p^*$, we have (see [2, eq. (4.13)])

$$\begin{aligned} |T(x+b, x)| &= |1 - \mathrm{e}^{\mathrm{i}(\theta_p(x+b) - \theta_p(x))}||P_\lambda(x+b, x)| \\ &\leq C \frac{|b|}{1 + |x - p|} |P_\lambda(x+b, x)| \leq C(1 + |p|) \frac{|b|}{1 + |x|} |P_\lambda(x+b, x)| . \end{aligned}$$

(Here and in the sequel, C denotes a generic constant, whose value is independent of any lattice sites in the given inequality, though that value may change from line to line.)

Since (1.2) holds for $g(H_B) = P_\lambda$, we have

$$|P_\lambda(x+b,x)| \ \le \ C_2(1+|x|)^\nu \mathrm{e}^{-\mu|b|} \ ,$$

but we also have $|P_\lambda(x+b,x)| \le 1$, because $\|P_\lambda\| \le 1$. Combing these two estimates gives

$$|P_\lambda(x+b,x)| \ \le \ \begin{cases} 1 & |b| \le \frac{2\nu}{\mu}\ln(|x|+1)\ , \\ C_2\,\mathrm{e}^{-\frac{\mu}{2}|b|} & |b| > \frac{2\nu}{\mu}\ln(|x|+1)\ . \end{cases} \tag{A.10}$$

Thus

$$\begin{aligned} &\left(\sum_x |T(x+b,x)|^3\right)^{1/3} \\ &\le \ C(1+|p|)|b| \left(\sum_{|x|<\mathrm{e}^{\frac{\mu}{2\nu}|b|}-1} \frac{\left[C_2 \mathrm{e}^{-\frac{\mu}{2}|b|}\right]^3}{(1+|x|)^3} \ + \sum_{|x|\ge \mathrm{e}^{\frac{\mu}{2\nu}|b|}-1} \frac{1}{(1+|x|)^3} \right)^{1/3} \\ &\qquad\qquad \le \ C(1+|p|)|b| \left(\mathrm{e}^{-\frac{\mu}{2}|b|} \ + \ \mathrm{e}^{-\frac{\mu}{6\nu}|b|}\right) \ . \end{aligned}$$

Since the last line is clearly summable over b, we see that $(U_p P_\lambda U_p^* - P)^3$ is trace class, and therefore the index N_p is well defined.

Turning now to part (ii), we note that we may just treat the case $p = -(\frac{1}{2},\frac{1}{2})$, $a = (a_1, 0)$, the case of translation in the 2-direction being similar. By (A.1, A.2, 2.2) we need to show that

$$\begin{aligned} &\mathrm{tr}(P_\lambda(\Delta\Lambda_1)P_\lambda^\perp \Lambda_2 P_\lambda) - \mathrm{tr}(P_\lambda \Lambda_2 P_\lambda^\perp(\Delta\Lambda_1)P_\lambda) \\ &\qquad = \ \mathrm{tr}(P_\lambda(\Delta\Lambda_1)P_\lambda^\perp \Lambda_2 P_\lambda) - \mathrm{tr}(P_\lambda^\perp(\Delta\Lambda_1)P_\lambda \Lambda_2 P_\lambda^\perp) \end{aligned} \tag{A.11}$$

vanishes, where $\Delta\Lambda_1(x) = \Lambda(x_1) - \Lambda(x_1 - a_1)$ is compactly supported in x_1. We claim that $(\Delta\Lambda_1)P_\lambda^\perp\Lambda_2 P_\lambda \in \mathfrak{I}_1$. This follows like (A.2) through the factorization

$$(\Delta\Lambda_1)P_\lambda^\perp \Lambda_2 P_\lambda \ = \ (\Delta\Lambda_1)\mathrm{e}^{3\delta|x_1|}\mathrm{e}^{-\delta|x|}\cdot \mathrm{e}^{-\delta|x|}\cdot \mathrm{e}^{-\delta|x|}\mathrm{e}^{3\delta|x_2|}P_\lambda^\perp \Lambda_2 P_\lambda \ ,$$

by noticing that the first factor, which is new, is bounded. Likewise

$$(\Delta\Lambda_1)P_\lambda\Lambda_2 P_\lambda^\perp \in \mathfrak{I}_1 \ .$$

Therefore (A.11) equals

$$\mathrm{tr}(\Delta\Lambda_1)P_\lambda^\perp\Lambda_2 P_\lambda - \mathrm{tr}(\Delta\Lambda_1)P_\lambda \Lambda_2 P_\lambda^\perp \ = \ \mathrm{tr}(\Delta\Lambda_1)\,[\Lambda_2, P_\lambda] \ = \ 0\ ,$$

by evaluating the trace in the position basis. □

The proof of Prop. 3 is now completed by the following result, with the translation invariance required in the argument of [5] now provided by Lemma 7.

Lemma 8. *Let $\Lambda_L = \{-L, \dots, L\}^2 \subset \mathbb{Z}^2$. Then*

$$\left.\begin{array}{c} N/2\pi \\ \sigma_B(\lambda) \end{array}\right\} = \lim_{L\to\infty} \frac{-2\mathrm{i}}{(2L+1)^2} \sum_{\substack{y,z\in\mathbb{Z}^2 \\ x\in\Lambda_L}} P_\lambda(x,y)P_\lambda(y,z)P_\lambda(z,x)\,\mathrm{Area}(x,y,z)\ , \tag{A.12}$$

where N, resp. $\sigma_B(\lambda)$ are the translation invariant values of N_p, resp. σ_p, and $\mathrm{Area}(x,y,z)$ *is the triangle's oriented area, namely $\frac{1}{2}(x-y)\wedge(y-z)$.*

Remark 3. The r.h.s. of (A.12) is the trace per unit volume of

$$-\mathrm{i}P_\lambda\left[[P_\lambda, X_1], [P_\lambda, X_2]\right]\ ,$$

which may be interpreted as the macroscopic version of (1.4).

Proof. The first statement makes use of Connes' area formula [9] in the version [5] adapted to the lattice [2]:

> For a fixed triplet $u^{(1)}$, $u^{(2)}$, $u^{(3)} \in \mathbb{Z}^2$, let $\alpha_i(p) \in (-\pi, \pi)$ be the angle of view from $p \in \mathbb{Z}^{2*}$ of $u^{(i+2)}$ relative to $u^{(i+1)}$ (with $\alpha_i(p) = 0$ if p lies between them). Then
>
> $$\sum_{p\in\mathbb{Z}^{2*}} \sum_{i=1}^{3} \sin\alpha_i(p) = 2\pi\,\mathrm{Area}(u^{(1)}, u^{(2)}, u^{(3)})\ . \tag{A.13}$$

By the computation of [5],

$$N_p = \mathrm{tr}(U_pP_\lambda U_p - P_\lambda)^3 = -2\mathrm{i} \sum_{x,y,z\in\mathbb{Z}^2} P_\lambda(x,y)P_\lambda(y,z)P_\lambda(z,x)S(p,x,y,z)\ .$$

with $S(p,x,y,z) = \sin\angle(x,p,y) + \sin\angle(y,p,z) + \sin\angle(z,p,x)$. Letting $\Lambda_L^* = \left\{-L+\frac{1}{2}, \dots, L+\frac{1}{2}\right\}^2 \subset \mathbb{Z}^{2*}$ we have that $N(2L+1)^2$ is the sum of the r.h.s. over $p \in \Lambda_L^*$.

We would like to replace the sum over $x \in \mathbb{Z}^2$, $p \in \Lambda_L^*$ by that over $x \in \Lambda_L$, $p \in \mathbb{Z}^{2*}$. The error is estimated by

$$\sum_{\substack{x\in\mathbb{Z}^2\setminus\Lambda_L \\ p\in\Lambda_L^*}} |f(p,x)| + \sum_{\substack{x\in\Lambda_L \\ p\in\mathbb{Z}^{2*}\setminus\Lambda_L^*}} |f(p,x)|\ , \tag{A.14}$$

where

$$f(p,x) := -2\mathrm{i} \sum_{y,z\in\mathbb{Z}^2} P_\lambda(x,y)P_\lambda(y,z)P_\lambda(z,x)S(p,x,y,z)\ .$$

By (1.2) for $g(H_B) = P_\lambda$ the points y, z are exponentially clustered around x, so we have $|f(p,x)| \le C_x(1+|p-x|)^{-3}$. However because of the pre-factor $(1+|x|)^\nu$ in (1.2), the constant C_x carries some dependence on x (as indicated), which must be controlled in order to bound (A.14).

In fact, the following estimate for $|f(p,x)|$ is true:

$$|f(p,x)| \le C\frac{[1+\ln(1+|x|)]^5}{1+|x-p|^3}\ . \tag{A.15}$$

Before proving (A.15), let us see how it allows us to complete the proof. Indeed, since

$$\sum_{\substack{x\in\Lambda_L\\ p\in\mathbb{Z}^{2*}\setminus\Lambda_L^*}} \frac{1}{(1+|x-p|)^3} \ = \ \mathcal{O}\left(L\ln L\right) \ , \quad L\to\infty \ ,$$

as far as the second term of (A.14) is concerned, we have

$$\sum_{\substack{x\in\Lambda_L\\ p\in\mathbb{Z}^{2*}\setminus\Lambda_L^*}} |f(p,x)| \ \le \ C[\ln L]^5 \sum_{\substack{x\in\Lambda_L\\ p\in\mathbb{Z}^{2*}\setminus\Lambda_L^*}} \frac{1}{(1+|x-p|)^3} \ = \ \mathcal{O}(L[\ln L]^6) \ .$$

For the first term we note that

$$[1+\ln(1+|x|)]^5 \ \le \ C(\ln L)^5[1+\ln(1+|x-p|)]^5 \ ,$$

for x,p in the indicated range and large L, resulting in

$$\sum_{\substack{x\in\mathbb{Z}^2\setminus\Lambda_L\\ p\in\Lambda_L^*}} |f(p,x)| \ \le \ C[\ln L]^5 \sum_{\substack{p\in\Lambda_L^*\\ x\in\mathbb{Z}^2\setminus\Lambda_L}} \frac{[1+\ln(1+|x-p|)]^5}{(1+|x-p|)^3} \ = \ \mathcal{O}(L[\ln L]^{11}) \ .$$

Therefore,

$$\begin{aligned} N(2L+1)^2 \ &= \ \sum_{\substack{x\in\Lambda_L\\ p\in\mathbb{Z}^{2*}}} f(p,x) \ + \ \mathcal{O}(L[\ln L]^{11}) \\ &= \ -2\mathrm{i}\sum_{\substack{x\in\Lambda_L\\ y,z\in\mathbb{Z}^2}} P_\lambda(x,y)P_\lambda(y,z)P_\lambda(z,x)\sum_{p\in\mathbb{Z}^{2*}} S(p,x,y,z) \ + \ \mathcal{O}(L[\ln L]^{11}) \ , \end{aligned}$$

which gives (A.12) for $N/2\pi$ after applying Connes' area formula and taking the limit $L\to\infty$.

As for the proof of (A.15), we consider separately the cases (i) $|p-x| < \frac{2\nu}{\mu}\ln(|x|+1)$ and (ii) $|p-x|\ge\frac{2\nu}{\mu}\ln(|x|+1)$. In case (i), we use the bound $|S(p,x,y,z)|\le 3$ to conclude

$$|f(p,x)| \ \le \ 6\sum_{y,z\in\mathbb{Z}^2} |P_\lambda(x,y)P_\lambda(y,z)P_\lambda(z,x)| \ \le \ 6\sum_{y\in\mathbb{Z}^2} |P_\lambda(x,y)| \ ,$$

since

$$\begin{aligned} \sum_{z\in\mathbb{Z}^2} |P_\lambda(y,z)P_\lambda(z,x)| \ &\le \ \left(\sum_{z\in\mathbb{Z}^2} |P_\lambda(y,z)|^2 \sum_{z\in\mathbb{Z}^2} |P_\lambda(z,x)|^2\right)^{1/2} \\ &\le \ [P_\lambda(y,y)P_\lambda(x,x)]^{1/2} \ \le \ 1 \ . \end{aligned}$$

Now by (A.10),

$$\sum_{y\in\mathbb{Z}^2}|P_\lambda(x,y)| \;\le\; \left(4\frac{\nu}{\mu}\ln(|x|+1)+1\right)^2 \;+\; C_2\sum_{|b|>\frac{2\nu}{\mu}\ln(|x|+1)} \mathrm{e}^{-\frac{\mu}{2}|b|}$$
$$\le\; C\,[1+\ln(|x|+1)]^2 \;\le\; C\frac{[1+\ln(|x|+1)]^5}{(1+|x-p|)^3}\;,$$

where in the last step we have used that $|x-p|\le\frac{2\nu}{\mu}\ln(|x|+1)$. This implies (A.15) in case (i). To prove (A.15) in case (ii), consider separately the contributions to $f(p,x)$ coming when both y and z fall inside the ball of radius $|p-x|$ around x and when one of y or z falls outside the ball. The latter contribution is exponentially small in $|x-p|$, since it is bounded by

$$6\left[\sum_{\substack{|y-x|\ge|p-x|\\ z\in\mathbb{Z}^2}} + \sum_{\substack{|z-x|\ge|p-x|\\ y\in\mathbb{Z}^2}}|P_\lambda(x,y)P_\lambda(y,z)P_\lambda(z,x)|\right]$$
$$\le\; 12\sum_{|y-x|\ge|p-x|}|P_\lambda(x,y)| \;\le\; C\mathrm{e}^{-\frac{\mu}{2}|x-p|}\;,$$

where in the last step we have used (A.10) and the fact that $|x-p|>\frac{2\nu}{\mu}\ln(|x|+1)$. To bound the former contribution note that in this case both $|\angle(y,p,x)|$ and $|\angle(z,p,x)|$ are smaller than $\frac{\pi}{2}$, and make use of the following estimates: (1) given $\alpha,\beta\in(-\frac{\pi}{2},\frac{\pi}{2})$,

$$|\sin\alpha+\sin\beta-\sin(\alpha+\beta)| \;\le\; |\sin\alpha|^3+|\sin\beta|^3\;,$$

and (2) given y with $|y-x|<|p-x|$,

$$|\sin\angle(y,p,x)| \;\le\; \frac{|y-x|}{1+|p-x|}\;.$$

Putting these two estimates together gives the following bound for the contribution with y,z in the ball of radius $|x-p|$ around x

$$\frac{C}{(1+|p-x|)^3}\sum_{|y-x|,|z-x|<|p-x|}|P_\lambda(x,y)P_\lambda(y,z)P_\lambda(z,x)|\left(|y-x|^3+|z-x|^3\right)$$
$$\le\frac{C}{(1+|p-x|)^3}\sum_{y\in\mathbb{Z}^2}|P_\lambda(x,y)|\,|y-x|^3 \;\le\; C\frac{[1+\ln(|x|+1)]^5}{(1+|p-x|)^3}\;,$$

where in the last step we have used (A.10). This proves (A.15) in case (ii) and completes the proof of (A.12) for $N/2\pi$.

The proof for σ_B is similar. By evaluating (A.7) in the position basis as in [5] we obtain

$$\sigma_p \;=\; \mathrm{i}\sum_{x,y,z\in\mathbb{Z}^2}P_\lambda(x,y)P_\lambda^\perp(y,z)P_\lambda(z,x)\cdot$$
$$\cdot\left[(\Lambda(y_1-p_1)-\Lambda(x_1-p_1))(\Lambda(z_2-p_2)-\Lambda(y_2-p_2))-(1\leftrightarrow 2)\right]\;. \quad \text{(A.16)}$$

We then sum over $p \in \Lambda_L^*$ and move the anchor from p to x (in this case the corresponding $f(p,x)$ decays exponentially in $|p-x|$, again with logarithmic growth in $|x|$). The sum over $p \in \mathbb{Z}^{2*}$ of the square bracket in (A.16) involves

$$\sum_{p_i \in \mathbb{Z}^*} (\Lambda(y_i - p_i) - \Lambda(x_i - p_i)) \;=\; x_i - y_i$$

and thus equals $(x_1 - y_1)(y_2 - z_2) - (x_2 - y_2)(y_1 - z_1) \,=\, 2\,\mathrm{Area}(x,y,z)$. The proof is completed by $P_\lambda^\perp(y,z) = \delta_{yz} - P_\lambda(y,z)$. $\square$

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